Lecture 13

Let A be an open set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a continuous function. For the moment, we assume that $f \geq 0$. Let $D \subseteq A$ be a compact and rectifiable set. Then $f|D$ is bounded, so $\int_D f$ is well-defined. Consider the set of all such integrals:

$$\# = \{\int_D f : D \subseteq A, D \text{ compact and rectifiable}\}. \tag{3.122}$$

Definition 3.22. The *improper integral of f over A* exists if $*$ is bounded, and we define the improper integral of f over A to be its l.u.b.

$$\int_A^{\#} f \equiv \text{l.u.b.} \int_D f = \text{ improper integral of } f \text{ over } A. \tag{3.123}$$

Claim. *If A is rectifiable and $f : A \to \mathbb{R}$ is bounded, then*

$$\int_A^{\#} f = \int_A f. \tag{3.124}$$

Proof. Let $D \subseteq A$ be a compact and rectifiable set. So,

$$\int_D f \leq \int_A f \tag{3.125}$$

$$\implies \sup_D \int_D f \leq \int_A f \tag{3.126}$$

$$\implies \int_A^{\#} f \leq \int_A f. \tag{3.127}$$

The proof of the inequality in the other direction is a bit more complicated.

Choose a rectangle Q such that $\bar{A} \subseteq \text{Int } Q$. Define $f_A : Q \to \mathbb{R}$ by

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{3.128}$$

By definition,

$$\int_A f = \int_Q f_A. \tag{3.129}$$

Now, let P be a partition of Q, and let $R_1, \ldots, R_k$ be rectangles belonging to a partition of A. If R is a rectangle belonging to P not contained in A, then $R - A \neq \phi$. In such a case, $m_R(f_A) = 0$. So

$$L(f_A, P) = \sum_{i=1}^{k} m_{R_i}(f_A) v(R_i). \tag{3.130}$$

On the rectangle R_i,

$$f_A = f \geq m_{R_i}(f_A). \tag{3.131}$$

So,

$$\begin{aligned} \sum_{i=1}^{k} m_{R_i}(f_A) v(R_i) &\leq \sum \int_{R_i} f \\ &= \int_D f \\ &\leq \int_A^{\#}, \end{aligned} \tag{3.132}$$

where $D = \bigcup R_i$, which is compact and rectifiable.

The above was true for all partitions, so

$$\int_Q f_A \leq \int_Z^{\#} f. \tag{3.133}$$

We proved the inequality in the other direction, so

$$\int_A f = \int_A^{\#} f. \tag{3.134}$$

$\square$

3.8 Exhaustions

Definition 3.23. A sequence of compact sets $C_i, i = 1, 2, 3 \ldots$ is an *exhaustion of A* if $C_i \subseteq \text{Int } C_{i_1}$ for every i, and $\bigcup C_i = A$.

It is easy to see that

$$\bigcup \text{Int } C_i = A. \tag{3.135}$$

Let C_i, $i = 1, 2, 3, \ldots$ be an exhaustion of A by compact rectifiable sets. Let $f : A \to \mathbb{R}$ be continuous and assume that $f \geq 0$. Note that

$$\int_{C_i} f \leq \int_{C_{i=1}} f, \tag{3.136}$$

since $C_{i=1} \supset C_i$. So

$$\int_{C_i} f, \ i = 1, 2, 3 \ldots \tag{3.137}$$

is an increasing (actually, non-decreasing) sequence. Hence, either $\int_{C_i} f \to \infty$ as $i \to \infty$, or it has a finite limit (by which we mean $\lim_{i\to\infty} \int_{C_i} f$ exists).

Theorem 3.24. *The following two properties are equivalent:*

1. $\int_A^\# f$ *exists,*
2. $\lim_{i\to\infty} \int_{C_i} f$ *exists.*

Moreover, if either (and hence both) property holds, then

$$\int_A^\# f = \lim_{i\to\infty} \int_{C_i} f. \tag{3.138}$$

Proof. The set C_i is a compact and rectifiable set contained in A. So, if

$$\int_A^\# f \text{ exists, then} \tag{3.139}$$

$$\int_{C_i} f \leq \int_A^\# f. \tag{3.140}$$

That shows that the sets

$$\int_{C_i} f,\ i = 1, 2, 3 \ldots \tag{3.141}$$

are bounded, and

$$\lim_{i\to\infty} \int_{C_i} f \leq \int_A^\# f. \tag{3.142}$$

Now, let us prove the inequality in the other direction.

The collection of sets $\{\text{Int } C_i : i = 1, 2, 3 \ldots\}$ is an open cover of A. Let $D \subseteq A$ be a compact rectifiable set contained in A. By the H-B Theorem,

$$D \subseteq \bigcup_{i=1}^N \text{Int } C_i, \tag{3.143}$$

for some N. So, $D \subseteq \text{Int } C_N \subseteq C_N$. For all such D,

$$\int_D f \leq \int_{C_i} f \leq \lim_{i\to\infty} \int_{C_i} f. \tag{3.144}$$

Taking the infimum over all D, we get

$$\int_A^\# f \leq \lim_{i\to\infty} \int_{C_i} f. \tag{3.145}$$

We have proved the inequality in both directions, so

$$\int_A^\# f = \lim_{i\to\infty} \int_{C_i} f. \tag{3.146}$$

□

A typical illustration of this theorem is the following example.

Consider the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}, \tag{3.147}$$

which we wrote in the normal integral notation from elementary calculus. In *our* notation, we would write this as

$$\int_{(0,1)} \frac{1}{\sqrt{x}}. \tag{3.148}$$

Let $C_N = [\frac{1}{N}, 1 - \frac{1}{N}]$. Then

$$\begin{aligned} \int_{(0,1)}^{\#} \frac{1}{\sqrt{x}} &= \lim_{N\to\infty} \int_{C_N} \frac{A}{\sqrt{x}} \\ &= 2\sqrt{x}|_{1/N}^{1-1/N} \to 2 \text{ as } N \to \infty. \end{aligned} \tag{3.149}$$

So,

$$\int_{(0,1)}^{\#} \frac{1}{\sqrt{x}} = 2. \tag{3.150}$$

Let us now remove the assumption that $f \geq 0$. Let $f : A \to \mathbb{R}$ be any continuous function on A. As before, we define

$$f_+(x) = \max\{f(x), 0\}, \tag{3.151}$$

$$f_-(x) = \max\{-f(x), 0\}. \tag{3.152}$$

We can see that f_+ and f_- are continuous.

Definition 3.25. The improper R. integral of f over A exists if and only if the improper R. integral of f_+ and f_- over A exist. Moreover, we define

$$\int_A^{\#} f = \int_A^{\#} f_+ - \int_A^{\#} f_-. \tag{3.153}$$

We compute the integral using an exhaustion of A.

$$\begin{aligned} \int_A^{\#} f &= \lim_{N\to\infty} \left(\int_{C_N} f_+ - \int_{C_N} f_- \right) \\ &= \lim_{N\to\infty} \int_{C_N} f. \end{aligned} \tag{3.154}$$

Note that $|f| = f_+ + f_-$, so

$$\lim_{N\to\infty} \left(\int_{C_N} f_+ + \int_{C_N} f_- \right) = \lim_{N\to\infty} \int_{C_N} |f|. \tag{3.155}$$

Therefore, the improper integral of f exists if and only if the improper integral of $|f|$ exists.

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases} \tag{3.156}$$

This is a $\mathcal{C}^\infty(\mathbb{R})$ function. Clearly, $f'(x) = f''(x) = \ldots = 0$ when $x = 0$, so in the Taylor series expansion of f at zero,

$$\sum a_n x^n = 0, \tag{3.157}$$

all of the coefficients a_n are zero. However, f has a non-zero value in every neighborhood of zero.

Take $a \in \mathbb{R}$ and $\epsilon > 0$. Define a new function $g_{a,a+\epsilon} : \mathbb{R} \to \mathbb{R}$ by

$$g_{a,a+\epsilon}(x) = \frac{f(x-a)}{f(x-a) + f(a+\epsilon-x)}. \tag{3.158}$$

The function $g_{a,a+\epsilon}$ is a $\mathcal{C}^\infty(\mathbb{R})$ function. Notice that

$$g_{a,a+\epsilon} = \begin{cases} 0 & \text{if } x \leq a, \\ 1 & \text{if } x \geq a+\epsilon. \end{cases} \tag{3.159}$$

Take b such that $a < a+\epsilon < b-\epsilon < b$. Define a new function $h_{a,b} \in \mathcal{C}^\infty(\mathbb{R})$ by

$$h_{a,b}(x) = g_{a,a+\epsilon}(x)(1 - g_{a-\epsilon,b}(x)). \tag{3.160}$$

Notice that

$$h_{a,b} = \begin{cases} 0 & \text{if } x \leq a, \\ 1 & \text{if } a+\epsilon \leq x \leq b-\epsilon, \\ 0 & \text{if } b \leq x. \end{cases} \tag{3.161}$$