Lecture 14

As before, let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases} \tag{3.162}$$

This is a $Cinf(\mathbb{R})$ function. Take the interval $[a, b] \in \mathbb{R}$ and define the function $f_{a,b} : \mathbb{R} \to \mathbb{R}$ by $f_{a,b}(x) = f(x - a)f(b - x)$. Note that $f_{a,b} > 0$ on (a, b), and $f_{a,b} = 0$ on $\mathbb{R} - (a, b)$.

We generalize the definition of f to higher dimensions. Let $Q \subseteq \mathbb{R}^n$ be a rectangle, where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Define a new map $f_Q : \mathbb{R}^n \to \mathbb{R}$ bye

$$f_Q(x_1, \ldots, x_n) = f_{a_1,b_1}(x_1) \ldots f_{a_n,b_n}(x_n). \tag{3.163}$$

Note that $f_Q > 0$ on Int Q, and that $f_Q = 0$ on $\mathbb{R}^n -$ Int Q.

3.9 Support and Compact Support

Now for some terminology. Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a continuous function.

Definition 3.26. The *support* of f is

$$\text{supp } f = \overline{\{x \in U : f(x) \neq 0\}}. \tag{3.164}$$

For example, supp $f_Q = Q$.

Definition 3.27. Let $f : U \to \mathbb{R}$ be a continuous function. The function f is *compactly supported* if supp f is compact.

Notation.

$$\mathcal{C}_0^k(U) = \text{The set of compactly supported } \mathcal{C}^k \text{ functions on } U. \tag{3.165}$$

Suppose that $f \in \mathcal{C}_0^k(U)$. Define a new set $U_1 = (\mathbb{R}^n - \text{supp } f)$. Then $U \cup U_1 = \mathbb{R}^n$, because supp $f \subseteq U$.

Define a new map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f} = \begin{cases} f & \text{on } U, \\ 0 & \text{on } U_1. \end{cases} \tag{3.166}$$

The function $\tilde{f}$ is $\mathcal{C}^k$ on U and $\mathcal{C}^k$ on U_1, so $\tilde{f}$ is in $\mathcal{C}_0^k(\mathbb{R}^n)$.

So, whenever we have a function $f \in \mathcal{C}^k$ is compactly supported on U, we can drop the tilde and think of f as in $\mathcal{C}_0^k(\mathbb{R}^n)$.

3.10 Partitions of Unity

Let $\{U_\alpha : \alpha \in I\}$ be a collection of of open subsets of $\mathbb{R}^n$ such that $U = \cup_\alpha U_\alpha$.

Theorem 3.28. *There exists a sequence of rectangles $Q_i,\ i = 1, 2, 3, \ldots$ such that*

1. Int $Q_i,\ i = 1, 2, 3\ldots$ *is a cover of U,*
2. *Each $Q_i \subset I_\alpha$ for some α,*
3. *For every point $p \in U$, there exists a neighborhood U_p of p such that $U_p \cap Q_i = \phi$ for all $i > N_p$.*

Proof. Take an exhaustion $A_1, A_2, A_3, \ldots$ of U. By definition, the exhaustion satisfies

$$\begin{cases} A_i \subseteq \text{Int } A_{i+1} \\ A_i \text{ is compact} \\ \cup A_i = U. \end{cases}$$

We previously showed that you can always find an exhaustion.

Let $B_i = A_i - \text{Int } A_{i-1}$. For each $x \in B_i$, let Q_x be a rectangle with $x \in \text{Int } Q_x$ such that $Q_x \subseteq U_\alpha$, for some alpha, and $Q_x \subset \text{Int } A_{i+1} - A_{i-2}$. Then, the collection of sets $\{\text{Int } Q_x : x \in B_i\}$ covers B_i. Each set B_i is compact, so, by the H-B Theorem, there exists a finite subcover Int $Q_{x_r} \equiv \text{Int } Q_{i,r},\ r = 1, \ldots, N_i$.

The rectangles $Q_{i,r}, 1 \le r \le N_i, i = 1, 2, 3\ldots$ satisfy the hypotheses of the theorem, after relabeling the rectangles in linear sequence Q_1, Q_2, Q_3, etc. (you should check this). □

The following theorem is called the Partition of Unity Theorem.

Theorem 3.29. *There exist functions $f_i \subseteq \mathcal{C}_0^\infty(U)$ such that*

1. *$f_1 \ge 0$,*
2. supp $f_i \subseteq U_\alpha$*, for some α,*
3. *For every $p \in U$, there exists a neighborhood U_p of p such that $U_p \cup \text{supp } f_i = \phi$ for all $i > N_p$,*
4. *$\sum f_i = 1$.*

Proof. Let $Q_i, i = 1, 2, 3, \ldots$ be a collection of rectangles with the properties of the previous theorem. Then the functions $f_{Q_i}, i = 1, 2, 3, \ldots$ have all the properties presented in the theorem, except for property 4. We now prove the fourth property. We now that $f_{Q_i} > 0$ on Int Q_i, and $\{\text{Int } Q_i : i = 1, 2, 3, \ldots\}$ is a cover of U. So, for every $p \in U$, $f_{Q_i}(p) > 0$ for some i. So

$$\sum f_{Q_i} > 0. \tag{3.167}$$

We can divide by a nonzero number, so we can define

$$f_i = \frac{f_{Q_i}}{\sum_{i=1}^{\infty} f_{Q_i}}. \tag{3.168}$$

This new function satisfies property 4. Note that the infinite sum converges because the sum has only a finite number of nonzero terms. □