Lecture 15

We restate the partition of unity theorem from last time. Let $\{U_\alpha : \alpha \in I\}$ be a collection of open subsets of $\mathbb{R}^n$ such that

$$U = \bigcup_{\alpha \in I} U_\alpha. \tag{3.169}$$

Theorem 3.30. *There exist functions* $f_i \subseteq \mathcal{C}_0^\infty(U)$ *such that*

1. $f_1 \geq 0$,
2. $\text{supp } f_i \subseteq U_\alpha$, *for some* α,
3. *For every* $p \in U$, *there exists a neighborhood* U_p *of* p *such that* $U_p \cup \text{supp } f_i = \phi$ *for all* $i > N_p$,
4. $\sum f_i = 1$.

Remark. Property (4) makes sense because of property (3), because at each point it is a finite sum.

Remark. A set of functions satisfying property (4) is called a *partition of unity.*

Remark. Property (2) can be restated as "the partition of unity is subordinate to the cover $\{U_\alpha : \alpha \in I\}$."

Let us look at some typical applications of partitions of unity.

The first application is to improper integrals. Let $\phi : U \to \mathbb{R}$ be a continuous map, and suppose

$$\int_U \phi \tag{3.170}$$

is well-defined. Take a partition of unity $\sum f_i = 1$. The function $f_i\phi$ is continuous and compactly supported, so it bounded. Let $\text{supp } f_i \subseteq Q_i$ for some rectangle Q_i. Then,

$$\int_{Q_i} f_i\phi \tag{3.171}$$

is a well-defined R. integral. It follows that

$$\int_U f_i\phi = \int_{Q_i} f_i\phi. \tag{3.172}$$

It follows that

$$\int_U \phi = \sum_{i=1}^{\infty} \int_{Q_i} f_i\phi. \tag{3.173}$$

This is proved in Munkres.

The second application of partitions of unity involves *cut-off functions.*

Let $f_i \in \mathcal{C}_0^\infty(U)$, $i = 1, 2, 3, \dots$ be a partition of unity, and let $A \subseteq U$ be compact.

Lemma 3.31. *There exists a neighborhood U' of A in U and a number $N > 0$ such that $A \cup \text{supp } f_i = \phi$ for all $i > N$.*

Proof. For any $p \in A$, there exists a neighborhood U_p of p and a number N_p such that $U' \cup \text{supp } f_i = \phi$ for all $i > N_p$. The collection of all these U_p is a cover of A. By the H-B Theorem, there exists a finite subcover U_{p_i}, $i = 1, 2, 3, \dots$ of A. Take $U_p = \cup U_{p_i}$ and take $N = \max\{N_{p_i}\}$. □

We use this lemma to prove the following theorem.

Theorem 3.32. *Let $A \subseteq \mathbb{R}^n$ be compact, and let U be an open set containing A. There exists a function $f \in \mathcal{C}_0^\infty(U)$ such that $f \equiv 1$ (identically equal to 1) on a neighborhood $U' \subset U$ of A.*

Proof. Choose U' and N as in the lemma, and let

$$f = \sum_{i=1}^{N} f_i. \tag{3.174}$$

Then $\text{supp } f_i \cap U' = \phi$ for all $i > N$. So, on U',

$$f = \sum_{i=1}^{\infty} f_i = 1. \tag{3.175}$$

□

Such an f can be used to create cut-off functions. We look at an application.

Let $\phi : U \to \mathbb{R}$ be a continuous function. Define $\psi = f\phi$. The new function ψ is called a cut-off function, and it is compactly supported with $\text{supp } \phi \subseteq U$. We can extend the domain of ψ by defining $\psi = 0$ outside of U. The extended function $\psi : \mathbb{R}^n \to \mathbb{R}$ is still continuous, and it equals ϕ on a neighborhood of A.

We look at another application, this time to *exhaustion functions.*

Definition 3.33. Given an open set U, and a collection of compact subsets A_i $i = 1, 2, 3, \dots$ of U, the sets A_i form an *exhaustion of* U if $A_i \subseteq \text{Int } A_{i+1}$ and $\cup A_i = U$ (this is just a quick reminder of the definition of exhaustion).

Definition 3.34. A function $\phi \in \mathcal{C}^\infty(U)$ is an *exhaustion function* if

1. $\phi > 0$,
2. the sets $A_i = \phi^{-1}([0, 1])$ are compact.

Note that this implies that the $A_i's$ are an exhaustion.

We use the fact that we can always find a partition of unity to show that we can always find exhaustion functions.

Take a partition of unity $f_i \in \mathcal{C}^\infty(U)$, and define

$$\phi = \sum_{i=1}^{\infty} i f_i. \tag{3.176}$$

This sum converges because only finitely many terms are nonzero.

Consider any point

$$p \notin \bigcup_{j \leq i} \text{supp } f_j. \tag{3.177}$$

Then,

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} f_k(p) \\ &= \sum_{k>i} f_k(p), \end{aligned} \tag{3.178}$$

so

$$\begin{aligned} \sum_{\ell=1}^{\infty} \ell f_\ell(p) &= \sum_{\ell > i} \ell f_\ell \\ &\geq i \sum_{\ell > i} f_\ell \\ &= i. \end{aligned} \tag{3.179}$$

That is, if $p \notin \cup_{j\leq i} \text{supp } f_j$, then $f(p) > i$. So,

$$\phi^{-1}([0, i]) \subseteq \bigcup_{j \leq i} \text{supp } f_j, \tag{3.180}$$

which you should check yourself. The compactness of the r.h.s. implies the compactness of the l.h.s.

Now we look at problem number 4 in section 16 of Munkres. Let A be an arbitrary subset of $\mathbb{R}^n$, and let $g : A \to \mathbb{R}^k$ be a map.

Definition 3.35. The function g *is* $\mathcal{C}^k$ *on* A if for every $p \in A$, there exists a neighborhood U_p of p in $\mathbb{R}^n$ and a $\mathcal{C}^k$ map $g^p : U_p \to \mathbb{R}^k$ such that $g^p | U_p \cap A = g | U_p \cap A$.

Theorem 3.36. *If $g : A \to \mathbb{R}^k$ is $\mathcal{C}^k$, then there exists a neighborhood U of A in $\mathbb{R}^n$ and a $\mathcal{C}^k$ map $\tilde{g} : U \to \mathbb{R}^k$ such that $\tilde{g} = g$ on A.*

Proof. This is a very nice application of partition of unity. Read Munkres for the proof. □