Lecture 19

We begin with a review of tensors and alternating tensors.

We defined $\mathcal{L}^k(V)$ to be the set of k-linear maps $T : V^k \to \mathbb{R}$. We defined $e_1, \dots, e_n$ to be a basis of V and $e_1^*, \dots, e_n^*$ to be a basis of V^*. We also defined $\{e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*\}$ to be a basis of $\mathcal{L}^k(V)$, where $I = (i_1, \dots, i_k)$, $1 \leq i_r \leq n$ is a multi-index. This showed that $\dim \mathcal{L}^k = n^k$.

We defined the permutation operation on a tensor. For $\sigma \in S_n$ and $T \in \mathcal{L}^k$, we defined $T^\sigma \in \mathcal{L}^k$ by $T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$. Then we defined that T is *alternating* if $T^\sigma = (-1)^\sigma T$. We defined $\mathcal{A}^k = \mathcal{A}^k(V)$ to be the space of all alternating k-tensors.

We defined the alternating operator $\text{Alt} : \mathcal{L}^k \to \mathcal{A}^k$ by $\text{Alt}\,(T) = \sum (-1)^\sigma T^\sigma$, and we defined $\psi_I = \text{Alt}\,(e_I^*)$, where $I = (i_1, \dots, i_k)$ is a strictly increasing multi-index. We proved the following theorem:

Theorem 4.24. *The ψ_I's (where I is strictly increasing) are a basis for $\mathcal{A}^k(V)$.*

Corollary 6. *If $0 \leq k \leq n$, then*

$$\dim \mathcal{A}^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{4.69}$$

Corollary 7. *If $k > n$, then $\mathcal{A}^k = \{0\}$.*

We now ask what is the kernel of Alt? That is, for which $T \in \mathcal{L}^k$ is $\text{Alt}\,(T) = 0$?

Let $T \in \mathcal{L}^k$ be a decomposable k-tensor, $T = \ell_1 \otimes \cdots \otimes \ell_k$, where each $\ell_i \in V^*$.

Definition 4.25. The k-tensor T is *redundant* if $\ell_i = \ell_{i+1}$ for some $1 \leq i \leq k-1$.

We define

$$\mathcal{I}^k \equiv \text{Span}\,\{\text{ redundant } k\text{-tensors }\}. \tag{4.70}$$

Claim. *If $T \in \mathcal{I}^k$, then $\text{Alt}\,(T) = 0$.*

Proof. It suffices to prove this for $T = \ell_1 \otimes \cdots \otimes \ell_k$, where $\ell_1 = \ell_{i+1}$ (T is redundant). Let $\tau = \tau_{i,i+1} \in S_k$. So, $T^\tau = T$. But

$$\begin{aligned} \text{Alt}\,(T^\tau) &= (-1)^\tau \,\text{Alt}\,(T) \\ &= -\,\text{Alt}\,(T), \end{aligned} \tag{4.71}$$

so $\text{Alt}\,(T) = 0$. □

Claim. *Suppose that $T \in \mathcal{I}^k$ and $T' \in \mathcal{L}^m$. Then $T' \otimes T \in \mathcal{I}^{k+n}$ and $T \otimes T' \in \mathcal{I}^{k+m}$.*

Proof. We can assume that T and T' are both decomposable tensors.

$$T = \ell_1 \otimes \cdots \otimes \ell_k, \quad \ell_i = \ell_{i+1}, \tag{4.72}$$

$$T' = \ell'_1 \otimes \cdots \otimes \ell'_m, \tag{4.73}$$

$$T \otimes T' = \ell_1 \otimes \cdots \otimes \underbrace{\ell_i \otimes \ell_{i+1}}_{\text{a redundancy}} \otimes \cdots \otimes \ell_k \otimes \ell'_1 \otimes \cdots \otimes \ell'_m \tag{4.74}$$

$$\in \mathcal{I}^{k+m}. \tag{4.75}$$

A similar argument holds for $T' \otimes T$. $\square$

Claim. *For each $T \in \mathcal{L}^k$ and $\sigma \in S_k$, there exists some $w \in \mathcal{I}^k$ such that*

$$T = (-1)^\sigma T^\sigma + W. \tag{4.76}$$

Proof. In proving this we can assume that T is decomposable. That is, $T = \ell_1 \otimes \cdots \otimes \ell_k$.

We first check the case $k = 2$. Let $T = \ell_1 \otimes \ell_2$. The only (non-identity) permutation is $\sigma = \tau_{1,2}$. In this case, $T = (-1)^\sigma T^\sigma + W$ becomes $W = T + T^\sigma$, so

$$\begin{aligned} W &= T + T^\sigma \\ &= \ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1 \\ &= (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2 \\ &\in \mathcal{I}^2. \end{aligned} \tag{4.77}$$

We now check the case k is arbitrary. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ and $\sigma = \tau_1 \tau_2 \ldots \tau_r \in S_k$, where the τ_i's are elementary transpositions. We will prove that $W \in \mathcal{I}^k$ by induction on r.

- Case $r = 1$: Then $\sigma = \tau_{i,i+1}$, and

$$\begin{aligned} W &= T + T^\sigma \\ &= (\ell_1 \otimes \cdots \otimes \ell_k) + (\ell_1 \otimes \cdots \otimes \ell_k)^\sigma \\ &= \ell_1 \otimes \cdots \otimes \ell_{i-1} \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k \\ &\in \mathcal{I}^k, \end{aligned} \tag{4.78}$$

  because $(\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \in \mathcal{I}^k$.

- Induction step ($(r-1) \implies r$): Let $\beta = \tau_2 \ldots \tau_r$, and let $\tau = \tau_1$ so that $\sigma = \tau_1 \tau_2 \ldots \tau_r = \tau\beta$. Then

$$T^\sigma = (T^\beta)^\tau. \tag{4.79}$$

  By induction, we know that

$$T^\beta = (-1)^\beta T + W, \tag{4.80}$$

for some $W \in \mathcal{I}^k$. So,

$$\begin{aligned} T^\sigma &= (-1)^\beta T^\tau + W^\tau \\ &= (-1)^\beta (-1)^\tau T + W^\tau \\ &= (-1)^\sigma T + W^\tau, \end{aligned} \tag{4.81}$$

where $W^\tau = (-1)^\tau W + W' \in \mathcal{I}^k$.

□

Corollary 8. *For every $T \in \mathcal{L}^k$,*

$$\operatorname{Alt}(T) = k!T + W \tag{4.82}$$

for some $W \in \mathcal{I}^k$.

Proof.

$$\operatorname{Alt}(T) = \sum_\sigma (-1)^\sigma T^\sigma, \tag{4.83}$$

but we know that $T^\sigma = (-1)^\sigma T + W_\sigma$, for some $W_\sigma \in \mathcal{I}^k$, so

$$\begin{aligned} \operatorname{Alt}(T) &= \sum_\sigma (T + (-1)^\sigma W_\sigma) \\ &= k!T + W, \end{aligned} \tag{4.84}$$

where $W = \sum_\sigma (-1)^\sigma W_\sigma \in \mathcal{I}^k$. □

Theorem 4.26. *Every $T \in \mathcal{L}^k$ can be written uniquely as a sum*

$$T = T_1 + T_2, \tag{4.85}$$

where $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$.

Proof. We know that $\operatorname{Alt}(T) = k!T + W$, for some $W \in \mathcal{I}^k$. Solving for T, we get

$$T = \underbrace{\frac{1}{k!}\operatorname{Alt}(T)}_{T_1} - \underbrace{\frac{1}{k!}W}_{T_2}. \tag{4.86}$$

We check uniqueness:

$$\operatorname{Alt}(T) = \underbrace{\operatorname{Alt}(T_1)}_{k!T_1} + \underbrace{\operatorname{Alt}(T_2)}_{0}, \tag{4.87}$$

so T_1 is unique, which implies that T_2 is also unique. □

Claim.

$$\mathcal{I}^k = \ker \operatorname{Alt}. \tag{4.88}$$

Proof. If Alt $T = 0$, then

$$T = -\frac{1}{k!}W, \quad W \in \mathcal{I}^k, \tag{4.89}$$

so $T \in \mathcal{I}^k$. □

The space $\mathcal{I}^k$ is a subspace of $\mathcal{L}^k$, so we can form the quotient space

$$\Lambda^k(V^*) \equiv \mathcal{L}^k/\mathcal{I}^k. \tag{4.90}$$

What's up with this notation $\Lambda^k(V^*)$? We motivate this notation with the case $k = 1$. There are no redundant 1-tensors, so $\mathcal{I}^1 = \{0\}$, and we already know that $\mathcal{L}^1 = V^*$. So

$$\Lambda^1(V^*) = V^*/\mathcal{I}^1 = \mathcal{L}^1 = V^*. \tag{4.91}$$

Define the map $\pi : \mathcal{L}^k \to \mathcal{L}^k/\mathcal{I}^k$. The map π is onto, and $\ker \pi = \mathcal{I}^k$.

Claim. *The map π maps $\mathcal{A}^k$ bijectively onto $\Lambda^k(V^*)$.*

Proof. Every element of Λ^k is of the form $\pi(T)$ for some $T \in \mathcal{L}^k$. We can write $T = T_1 + T_2$, where $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$. So,

$$\begin{aligned} \pi(T) &= \pi(T_1) + \pi(T_2) \\ &= \pi(T_1) + 0 \\ &= \pi(T_1). \end{aligned} \tag{4.92}$$

So, π maps $\mathcal{A}^k$ onto Λ^k. Now we show that π is one-to-one. If $T \in \mathcal{A}^k$ and $\pi(T) = 0$, then $T \in \mathcal{I}^k$ as well. We know that $\mathcal{A}^k \cap \mathcal{I}^k = \{0\}$, so π is bijective. □

We have shown that

$$\mathcal{A}^k(V) \cong \Lambda^k(V^*). \tag{4.93}$$

The space $\Lambda^k(V^*)$ is not mentioned in Munkres, but sometimes it is useful to look at the same space in two different ways.