Lecture 24

We review the pullback operation from last lecture. Let U be open in $\mathbb{R}^m$ and let V be open in $\mathbb{R}^n$. Let $f : U \to V$ be a $\mathcal{C}^\infty$ map, and let $f(p) = q$. From the map

$$df_p : T_p\mathbb{R}^m \to T_q\mathbb{R}^n, \tag{4.212}$$

we obtain the pullback map

$$\begin{aligned} (df_p)^* : \Lambda^k(T_q^*) &\to \Lambda^k(T_p^*) \\ \omega \in \Omega^k(V) &\to f^*\omega \in \Omega^k(U). \end{aligned} \tag{4.213}$$

We define, $f^*\omega_p = (df_p)^*\omega_q$, when $\omega_q \in \Lambda^k(T_q^*)$.

The pullback operation has some useful properties:

1. If $\omega_i \in \Omega^{k_i}(V), i = 1, 2$, then

$$f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2. \tag{4.214}$$

2. If $\omega \in \Omega^k(V)$, then

$$df^*\omega = f^*d\omega. \tag{4.215}$$

We prove some other useful properties of the pullback operation.

Claim. *For all $\omega \in \Omega^k(W)$,*

$$f^*g^*\omega = (g \circ f)^*\omega. \tag{4.216}$$

Proof. Let $f(p) = q$ and $g(q) = w$. We have the pullback maps

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*) \tag{4.217}$$

$$(dg_q)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_q^*) \tag{4.218}$$

$$(g \circ f)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*). \tag{4.219}$$

The chain rule says that

$$(dg \circ f)_p = (dg)_q \circ (df)_p, \tag{4.220}$$

so

$$d(g \circ f)_p^* = (df_p)^*(dg_q)^*. \tag{4.221}$$

□

Let U, V be open sets in $\mathbb{R}^n$, and let $f : U \to V$ be a $\mathcal{C}^\infty$ map. We consider the pullback operation on n-forms $\omega \in \Omega^n(V)$. Let $f(0) = q$. Then

$$(dx_i)_p, \quad i = 1, \dots, n, \quad \text{is a basis of } T_p^*, \text{ and} \tag{4.222}$$

$$(dx_i)_q, \quad i = 1, \dots, n, \quad \text{is a basis of } T_q^*. \tag{4.223}$$

Using $f_i = x_i \circ f$,

$$\begin{aligned}(df_p)^*(dx_i)_q &= (df_i)_p \\ &= \sum \frac{\partial f_i}{\partial x_j}(p)(dx_j)_p.\end{aligned} \tag{4.224}$$

In the Multi-linear Algebra notes, we show that

$$(df_p)^*(dx_1)_q \wedge \cdots \wedge (dx_n)_q = \det \left[\frac{\partial f_i}{\partial x_j}(p)\right] (dx_1)_p \wedge \cdots \wedge (dx_n)_p. \tag{4.225}$$

So,

$$f^* dx_1 \wedge \cdots \wedge dx_n = \det \left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n. \tag{4.226}$$

Given $\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n$, where $\phi \in \mathcal{C}^\infty$,

$$f^*\omega = \phi(f(x)) \det \left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n. \tag{4.227}$$

5 Integration with Differential Forms

Let U be an open set in $\mathbb{R}^n$, and let $\omega \in \Omega^k(U)$ be a differential k-form.

Definition 5.1. The *support* of ω is

$$\text{supp } \omega = \overline{\{p \in U : \omega_p \neq 0\}}. \tag{5.1}$$

Definition 5.2. The k-form ω is *compactly supported* if supp ω is compact. We define

$$\Omega^k_c(U) = \text{ the space of all compactly supported } k\text{-forms}. \tag{5.2}$$

Note that

$$\Omega^0_c(U) = \mathcal{C}^\infty_0(\mathbb{R}^n). \tag{5.3}$$

Given $\omega \in \Omega^n_c(U)$, we can write

$$\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n, \tag{5.4}$$

where $\phi \in \mathcal{C}^\infty_0(U)$.

Definition 5.3.

$$\int_U \omega \equiv \int_U \phi = \int_U \phi(x) dx_1 \dots dx_n. \tag{5.5}$$

We are going to state and prove the change of variables theorem for integrals of differential k-forms. To do so, we first need the notions of *orientation preserving* and *orientation reversing*.

Let U, V be open sets in $\mathbb{R}^n$. Let $f : U \to V$ be a $\mathcal{C}^\infty$ diffeomorphism. That is, for every $p \in U$, $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective. We associate $Df(p)$ with the matrix

$$Df(p) \cong \left[\frac{\partial f_i}{\partial x_j}(p)\right]. \tag{5.6}$$

The map f is a diffeomorphism, so

$$\det\left[\frac{\partial f_i}{\partial x_j}(p)\right] \neq 0. \tag{5.7}$$

So, if U is connected, then this determinant is either positive everywhere or negative everywhere.

Definition 5.4. The map f is *orientation preserving* if $\det > 0$ everywhere. The map f is *orientation reversing* if $\det < 0$ everywhere.

The following is the change of variables theorem:

Theorem 5.5. *If $\omega \in \Omega_c^n(V)$, then*

$$\int_U f^*\omega = \int_V \omega \tag{5.8}$$

if f is orientation preserving, and

$$\int_U f^*\omega = -\int_V \omega \tag{5.9}$$

if f is orientation reversing.

In Munkres and most texts, this formula is written in slightly uglier notation. Let $\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n$, so

$$f^*\omega = \phi(f(x)) \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n. \tag{5.10}$$

The theorem can be written as following:

Theorem 5.6. *If f is orientation preserving, then*

$$\int_V \phi = \int_U \phi \circ f \det\left[\frac{\partial f_i}{\partial x_j}\right]. \tag{5.11}$$

This is the coordinate version of the theorem.

We now prove a useful theorem found in the Supplementary Notes (and Spivak) called Sard's Theorem.

Let U be open in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a $\mathcal{C}^1(U)$ map. For every $p \in U$, we have the map $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$. We say that p is a *critical point of f* if $Df(p)$ is *not* bijective. Denote

$$C_f = \text{ the set of all critical points of } f. \tag{5.12}$$

Sard's Theorem. *The image $f(C_f)$ is of measure zero.*

Proof. The proof is in the Supplementary Notes. □

As an example of Sard's Theorem, let $c \in \mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be the map defined by $f(x) = c$. Note that $Df(p) = 0$ for all $p \in U$, so $C_f = U$. The set $C_f = U$ is not a set of measure zero, but $f(C_f) = \{c\}$ is a set of measure zero.

As an exercise, you should prove the following claim:

Claim. *Sard's Theorem is true for maps $f : U \to \mathbb{R}^n$, where U is an open, connected subset of $\mathbb{R}$.*

Proof Hint: Let $f \in \mathcal{C}^\infty(U)$ and define $g = \frac{\partial f}{\partial x}$. The map g is continuous because $f \in \mathcal{C}^1(U)$. Let $I = [a, b] \subseteq U$, and define $\ell = b - a$. The continuity of g implies that g is uniformly continuous on I. That is, for every $\epsilon > 0$, there exists a number $N > 0$ such that $|g(x) - g(y)| < \epsilon$ whenever $x, y \in I$ and $|x - y| < \ell/N$.

Now, slice I into N equal subintervals. Let $I_r, r = 1, \dots, k \leq N$ be the subintervals intersecting C_f. Prove the following lemma:

Lemma 5.7. *If $x, y \in I_r$, then $|f(x) - f(y)| < \epsilon\ell/N$.*

Proof Hint: Find $c \in I_r$ such that $f(x) - f(y) = (x - y)g(c)$. There exists $c_0 \in I_r \cap C_f$ if and only if $g(c_0) = 0$. So, we can take

$$|g(c)| = |g(c) - g(c_0)| \leq \epsilon. \tag{5.13}$$

Then $|f(x) - f(y)| \leq \epsilon\ell/N$. □

From the lemma, we can conclude that

$$f(I_r) \equiv J_r \tag{5.14}$$

is of length less than $\epsilon\ell/N$. Therefore,

$$f(C_f \cap I) \subset \bigcup_{r=1}^{k} J_r \tag{5.15}$$

is of length less than

$$\frac{\epsilon\ell}{N} k \leq \frac{\epsilon\ell N}{N} = \epsilon\ell. \tag{5.16}$$

Letting $\epsilon \to 0$, we find that $F(C_f \cap I)$ is of measure zero.

To conclude the proof, let $I_m, m = 1, 2, 3, \dots,$ be an exhaustion of U by closed intervals $I_1 \subset I_2 \subset I_3 \subset \cdots$ such that $\bigcup I_m = U$. We have shown that $f(C_f \cap I_m)$ is measure zero. So, $f(C_f) = \bigcup f(C_f \cap I_m)$ implies that $f(C_f)$ is of measure zero. $\square$