Lecture 27

We proved the following Poincare Lemma:

Poincare Lemma. *Let U be a connected open subset of $\mathbb{R}^n$, and let $\omega \in \Omega_c^n(U)$. The following conditions are equivalent:*

1. $\int_U \omega = 0$,
2. $\omega = d\mu$, *for some* $\mu \in \Omega_c^{n-1}(U)$.

We first proved this for the case $U = \text{Int } Q$, where Q was a rectangle. Then we used this result to generalize to arbitrary open connected sets. We discussed a nice application: proper maps and degree.

Let U, V be open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a $\mathcal{C}^\infty$ map. The map f is *proper* if for every compact set $C \subseteq V$, the pre-image $f^{-1}(C)$ is also compact. Hence, if f is proper, then

$$f^*\Omega_c^k(V) \subseteq \Omega_c^k(U). \tag{5.88}$$

That is, if $\omega \in \Omega_c^k(V)$, then $f^*\omega \in \Omega_c^k(U)$, for all k.

When $k = n$,

$$\omega \in \Omega_c^n(V). \tag{5.89}$$

In which case, we compare

$$\int_v \omega \quad \text{and} \quad \int_U f^*\omega. \tag{5.90}$$

Using the Poincare Lemma, we obtain the following theorem.

Theorem 5.14. *There exists a constant γ_f with the property that for all $\omega \in \Omega_c^n(V)$,*

$$\int_U f^*\omega = \gamma_f \int_V \omega. \tag{5.91}$$

We call this constant the *degree of f*,

Definition 5.15.

$$\deg(f) = \gamma_f. \tag{5.92}$$

Let U, V, W be open connected subsets of $\mathbb{R}^n$, and let $f : U \to V$ and $g : V \to W$ be proper $\mathcal{C}^\infty$ maps. Then the map $g \circ f : U \to W$ is proper, and

$$\deg(g \circ f) = \deg(f)\deg(g). \tag{5.93}$$

Proof Hint: For all $\omega \in \Omega_c^n(W)$, $(g \circ f)^*\omega = f^*(g^*\omega)$. $\square$

We give some examples of the degree of various maps. Let $f = T_a$, the transposition by a. That is, let $f(x) = x + a$. From #4 in section 4 of the Supplementary Notes, the map $T_a : \mathbb{R}^n \to \mathbb{R}^n$ is proper. One can show that $\deg(T_a) = 1$.

As another example, let $f = A : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective liner map. Then

$$\deg A = \begin{cases} 1 & \text{if } \det A > 0, \\ -1 & \text{if } \det A < o. \end{cases} \tag{5.94}$$

We now study the degree as it pertains to orientation preserving and orientation reversing maps.

Let U, V be connected open sets in $\mathbb{R}^n$, and let $f : U \to V$ be a diffeomorphism. Take $p \in U$. Then $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto. The map f is *orientation preserving* if $\det Df(p) > 0$ for all $p \in U$, and the map f is *orientation reversing* if $\deg Df(p) < 0$ for all $p \in U$.

Theorem 5.16. *If f is orientation preserving, then* $\deg(f) = 1$*; if f is orientation reversing, then* $\deg(f) = -1$.

Proof. Let $a \in U$ and $b = f(a)$. Define

$$f_{\text{old}} = f, \tag{5.95}$$

and define

$$f_{\text{new}} = T_{-b} \circ f_{\text{old}} \circ T_a, \tag{5.96}$$

where $T_{-b} : \mathbb{R}^n \to \mathbb{R}^n$ and $T_a : \mathbb{R}^n \to \mathbb{R}^n$ are transpositions by $-b$ and a, respectively.

By the formula $\deg(g \circ f) = \deg(f)\deg(g)$,

$$\begin{aligned} \deg(f_{\text{new}}) &= \deg(T_{-b}) \deg(f_{\text{old}}) \deg(T_a) \\ &= \deg(_{\text{old}}). \end{aligned} \tag{5.97}$$

By replacing f with f_{new}, we can assume that $0 \in U$ and $f(0) = 0$.

We can make yet another simplification, that $Df(0) = I$, the identity. To see this, let $Df(0) = A$, where $A : \mathbb{R}^n \to \mathbb{R}^n$. Taking our new f, we redefine $f_{\text{old}} = f$, and we redefine $f_{\text{new}} = A^{-1} \circ f_{\text{old}}$. Then,

$$\deg(f_{\text{new}}) = \deg(A) \deg \deg(f_{\text{old}}), \tag{5.98}$$

where

$$\begin{aligned} \deg A &= \deg(Df_{\text{old}}) \\ &= \begin{cases} 1 & \text{if } Df_{\text{old}} \text{ is orient. preserving,} \\ -1 & \text{if } Df_{\text{old}} \text{ is orient. reversing.} \end{cases} \end{aligned} \tag{5.99}$$

We again replace f with f_{new}. It suffices to prove the theorem for this new f. To summarize, we can assume that

$$0 \in U, \quad f(0) = 0. \quad \text{and } Df(0) = I. \tag{5.100}$$

Consider $g(x) = x - f(x)$ (so $f(x) = x - g(x)$). Note that $(Dg)(0) = I - I = 0$. If we write $g = (g_1, \dots, g_n)$, then

$$\left[\frac{\partial g_i}{\partial x_j}(0)\right] = 0. \tag{5.101}$$

So, each $\frac{\partial g_i}{\partial x_j}(0) = 0$.

Lemma 5.17. *There exists $\delta > 0$ such that for all $|x| < \delta$,*

$$|g(x)| \leq \frac{|x|}{2}. \tag{5.102}$$

Proof. So far, we know that $g(0) = 0 - f(0) = 0$, and $\frac{\partial g_i}{\partial x_j}(0) = 0$. By continuity, there exists $\delta > 0$ such that

$$\left|\frac{\partial g_i}{\partial x_j}(x)\right| \leq \frac{1}{2n}, \tag{5.103}$$

for all $|x| < \delta$. Using the Mean-value Theorem, for all $|x| < \delta$,

$$\begin{aligned} g_i(x) &= g_i(x) - g_i(0) \\ &= \sum \frac{\partial g_i}{\partial x_j}(c) x_j, \end{aligned} \tag{5.104}$$

where $c = t_0 x$ for some $0 < t_0 < 1$. So,

$$\begin{aligned} |g_i(x)| &\leq \sum_{i=1}^{n} \frac{1}{2n} |x_i| \\ &\leq \frac{1}{2} \max\{|x_i|\} \\ &= \frac{1}{2}|x|. \end{aligned} \tag{5.105}$$

□

Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ as follows. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, defined to have the follow properties

$$\rho(x) = \begin{cases} 1 & \text{if } |x| < \delta/2, \\ 0 & \text{if } |x| > \delta, \\ 0 \leq \rho(x) \leq 1 & \text{otherwise.} \end{cases} \tag{5.106}$$

Remember that $f(x) = x - g(x)$. Define

$$\tilde{f} = \begin{cases} x - \rho(x) g(x) & \text{if } |x| < \delta, \\ x & \text{if } |x| > \delta. \end{cases} \tag{5.107}$$

Claim. *The map $\tilde{f}$ has the following properties:*

1. $\tilde{f} = f(x)$ *for all* $|x| < \frac{\delta}{2}$,
2. $\tilde{f} = x$ *for all* $|x| > \delta$,
3. $|\tilde{f}(x)| \geq \frac{|x|}{2}$,
4. $|\tilde{f}(x)| \leq 2|x|$.

Proof. We only proof properties (3) and (4). First we prove property (3). We have $\tilde{f}(x) = x - \rho(x)g(x) = x$ when $|x| \geq \delta$, so $|\tilde{f}(x)| = |x|$ when $|x| \geq \delta$. For $|x| < \delta$, we have

$$\begin{aligned} |\tilde{f}(x)| &\geq |x| - \rho(x)|g(x)| \\ &= |x| - |g(x)| \\ &\geq |x| - \frac{|x|}{2} \\ &= \frac{|x|}{2}. \end{aligned} \tag{5.108}$$

We now prove property (4). We have $\tilde{f}(x) = x - \rho(x)g(x)$, so $|\tilde{f}(x)| = |x|$ for $x \geq \delta$. For $x < \delta$, we have

$$\begin{aligned} |\tilde{f}(x)| &\leq |x| + \rho(x)|g(x)| \\ &\leq |x| + \frac{1}{2}|x| \\ &\leq 2|x|. \end{aligned} \tag{5.109}$$

□

Let $Q_r \equiv \{x \in \mathbb{R}^n : |x| \leq r\}$. The student should check that

$$\text{Property (3)} \implies \tilde{f}^{-1}(Q_r) \subseteq Q_{2r} \tag{5.110}$$

and that

$$\text{Property (4)} \implies \tilde{f}^{-1}(\mathbb{R}^n - Q_{2r}) \subseteq \mathbb{R}^n - Q_r \tag{5.111}$$

Notice that $\tilde{f}^{-1}(Q_r) \subseteq Q_{2r} \implies \tilde{f}$ is proper.

Now we turn back to the map f. Remember that $f : U \to V$ is a diffeomorphism and that $f(0) = 0$. So, the set $f(\text{Int } Q_{\delta/2})$ is an open neighborhood of 0 in $\mathbb{R}^n$. Take

$$\omega \in \Omega^n_c(f(\text{Int } Q_{\delta/2}) \cap \text{Int } Q_{\delta/4}) \tag{5.112}$$

such that

$$\int_{\mathbb{R}^n} \omega = 1. \tag{5.113}$$

Then,

$$f^*\omega \in \Omega_c^n(Q_{\delta/2}) \tag{5.114}$$

and

$$\tilde{f}^*\omega \in \Omega_c^n(Q_{\delta/2}), \tag{5.115}$$

by Equation 5.110. This shows that $f^*\omega = \tilde{f}^*\omega$. Hence,

$$\begin{aligned} \int_U f^*\omega = \int_U \tilde{f}^*\omega = \deg(f) \int_V \omega \\ = \deg(\tilde{f}) \int_V \omega, \end{aligned} \tag{5.116}$$

where

$$\int_V \omega = 1. \tag{5.117}$$

Therefore,

$$\deg(f) = \deg(\tilde{f}). \tag{5.118}$$

Now, let us use Equation 5.111. Choose $\omega \in \Omega_c^n(\mathbb{R}^n - Q_{2\delta})$. So,

$$f^*\omega \in \Omega_c^n(\mathbb{R}^n - Q_\delta). \tag{5.119}$$

Again we take

$$\int_{\mathbb{R}^n} \omega = 1. \tag{5.120}$$

By property (2), $\tilde{f} = I$ on $\mathbb{R}^n - Q_\delta$, so

$$\tilde{f}^*\omega = \omega. \tag{5.121}$$

Integrating,

$$\int_{\mathbb{R}}^n \tilde{f}^*\omega = \deg(\tilde{f}) \int_{\mathbb{R}}^n \omega = \int_{\mathbb{R}}^n \omega. \tag{5.122}$$

Therefore,

$$\deg(f) = \deg(\tilde{f}) = 1. \tag{5.123}$$

□