Lecture 29

We have been studying the important invariant called the degree of f. Today we show that the degree is a "topological invariant."

5.3 Topological Invariance of Degree

Recall that given a subset A of $\mathbb{R}^m$ and a function $F : A \to \mathbb{R}^\ell$, we say that F is $\mathcal{C}^\infty$ if it extends to a $\mathcal{C}^\infty$ map on a neighborhood of A.

Let U be open in $\mathbb{R}^n$, let V be open in $\mathbb{R}^k$, and let $A = U \times [0,1]$.

Definition 5.22. Let $f_0, f_1 : U \to V$ be $\mathcal{C}^\infty$ maps. The maps f_0 and f_1 are *homotopic* if there is a $\mathcal{C}^\infty$ map $F : U \times [0,1] \to V$ such that $F(p,0) = f_0(p)$ and $F(p,1) = f_1(p)$ for all $p \in U$.

Let $f_t : U \to V$ be the map defined by

$$f_t(p) = F(p,t). \tag{5.144}$$

Note that $F \in \mathcal{C}^\infty \implies f_t \in \mathcal{C}^\infty$. So, $f_t : U \to V$, where $0 \leq t \leq 1$, gives a family of maps parameterized by t. The family of maps f_t is called a $\mathcal{C}^\infty$ *deformation* of f_0 into f_1.

Definition 5.23. The map F is a *proper homotopy* if for all compact sets $A \subseteq V$, the pre-image $F^{-1}(A)$ is compact.

Denote by π the map $\pi : U \times [0,1] \to U$ that sends $(p,t) \to t$. Let $A \subseteq V$ be compact. Then $B = \pi(F^{-1}(A))$ is compact, and for all t, $f_t^{-1}(A) \subseteq B$. As a consequence, each f_t is proper.

We concentrate on the case where U, V are open connected subsets of $\mathbb{R}^n$ and $f_0, f_1 : U \to V$ are proper $\mathcal{C}^\infty$ maps. We now prove that the degree is a topological invariant.

Theorem 5.24. *If f_0 and f_1 are homotopic by a proper homotopy, then*

$$\deg(f_0) = \deg(f_1). \tag{5.145}$$

Proof. Let $\omega \in \Omega^n_c(V)$ and let $\operatorname{supp} \omega = A$. Let $F : U \times I \to V$ be a proper homotopy between f_0 and f_1. Take $B = \pi(F^{-1}(A))$, which is compact. For all $t \in [0,1]$, $f_t^{-1}(A) \subseteq B$.

Let us compute $f_t^*\omega$. We can write $\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n$, where $\operatorname{supp} \phi \subseteq A$. So,

$$f_t^*\omega = \phi(F(x,t)) \det \left[\frac{\partial F_i}{\partial x_j}(x,t) \right] dx_1 \wedge \cdots \wedge dx_n, \tag{5.146}$$

and

$$\int_U f_t^* \omega = \deg(f_t) \int_V \omega. \\ = \int_U \phi(F(x,t)) \det \left[\frac{\partial F_i}{\partial x_j}(x,t) \right] dx_1 \dots dx_n. \tag{5.147}$$

Notice that the integrand is supported in the compact set B for all t, and it is $\mathcal{C}^\infty$ as a function of x and t. By Exercise #2 in section 2 of the Supplementary Notes, this implies that the integral is $\mathcal{C}^\infty$ in t. From Equation 5.147, we can conclude that $\deg(f_t)$ is a $\mathcal{C}^\infty$ function of t.

Now here is the trick. Last lecture we showed that $\deg(f_t)$ is an integer. Since $\deg(f_t)$ is continuous, it must be a constant $\deg(f_t) =$ constant. □

We consider a simple application of the above theorem. Let $U = V = \mathbb{R}^2$, and think of $\mathbb{R}^2 = \mathbb{C}$. We make the following associations:

$$i^2 = -1 \tag{5.148}$$

$$z = x + iy \tag{5.149}$$

$$\bar{z} = x - iy \tag{5.150}$$

$$z\bar{z} = |z|^2 = x^2 + y^2 \tag{5.151}$$

$$dz = dx + idy \tag{5.152}$$

$$d\bar{z} = dx - idy \tag{5.153}$$

$$dz \wedge d\bar{z} = -2idx \wedge dy \tag{5.154}$$

$$dx \wedge dy = \frac{1}{2} i dz \wedge d\bar{z}. \tag{5.155}$$

Consider a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, thinking of $\mathbb{R}^2 = \mathbb{C}$, defined by

$$f(z) = z^n + \sum_{i=0}^{n-1} c_i z^i, \ c_i \in \mathbb{C}. \tag{5.156}$$

Claim. *The map f is proper.*

Proof. Let $C = \sum |c_i|$. For $|z| > 1$,

$$\left| \sum_{i=0}^{n-1} c_i z^i \right| \leq C|z|^{n-1}. \tag{5.157}$$

So,

$$|f(z)| \geq |z|^n - \left| \sum c_i z^i \right| \\ = |z|^n - C|z|^{n-1} \\ = |z|^n \left(1 - \frac{C}{|z|} \right). \tag{5.158}$$

For $|z| > 2C$,

$$|f(z)| \geq \frac{|z|^n}{2}. \tag{5.159}$$

So, if $R > 1$ and $R > 2C$, then $f^{-1}(B_R) \subseteq B_{R_1}$, where $R_1^n/2 \leq R$ (and where B_r denotes the ball of radius r). So f is proper. □

Now, let us define a homotopy $F : \mathbb{C} \times [0,1] \to \mathbb{C}$ by

$$F(z,t) = z^n + t\sum_{i=0}^{n-1} c_i z^i. \tag{5.160}$$

We claim that $F^{-1}(B_R) \subseteq B_{R_1} \times [0,1]$, by exactly the same argument as above. So F is proper.

Notice that

$$F(z,1) = f_1(z) = f(z), \tag{5.161}$$

$$F(z,0) = f_0(z) = z^n. \tag{5.162}$$

So, by the above theorem, $\deg(f) = \deg(f_0)$.

Let us compute $\deg(f_0)$ by brute force. We have $f_0(z) = z^n$, so

$$f_0^* dz = dz^n = nz^{n-1}dz, \tag{5.163}$$

$$f_0^* d\bar{z} = d\bar{z}^n = n\bar{z}^{n-1}d\bar{z}. \tag{5.164}$$

Using the associations defined above,

$$\begin{aligned} f_0^*(dx \wedge dy) &= \frac{i}{2} f_0^*(dz \wedge d\bar{z}) \\ &= \frac{i}{2} f_0^* dz \wedge f_0^* d\bar{z} \\ &= \frac{i}{2} n^2 |z|^{2(n-1)} dz \wedge d\bar{z} \\ &+ n^2|z|^{2n-2} dx \wedge dy. \end{aligned} \tag{5.165}$$

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\int_0^\infty \phi(s)ds = 1. \tag{5.166}$$

Let $\omega = \phi(|z|^2)dx \wedge dy$. We calculate $\int_{\mathbb{R}^2} \omega$. Let us use polar coordinates, where

$r = \sqrt{x^2 + y^2} = |z|$.

$$\begin{aligned} \int_{\mathbb{R}^2} \omega &= \int_{\mathbb{R}^2} \phi(|z|^2) dx dy \\ &= \int_{\mathbb{R}^2} \phi(r^2) r dr d\theta \\ &= 2\pi \int_o^\infty \phi(r^2) r dr \\ &+ 2\pi \int_0^\infty \phi(s) \frac{ds}{2} \\ &= \pi. \end{aligned} \tag{5.167}$$

Now we calculate $\int f_0^* \omega$. First, we note that

$$f_0^* \omega = \phi(|z|^{2n}) n^2 |z|^{2n-2} dx \wedge dy. \tag{5.168}$$

So,

$$\begin{aligned} \int f_0^* \omega &= n^2 \int_0^\infty \phi(r^{2n}) r^{2n-2} r dr d\theta \\ &= n^2 (2\pi) \int_0^\infty \phi(r^{2n}) r^{2n-1} dr \\ &= n^2 (2\pi) \int_0^\infty \phi(s) \frac{ds}{2n} \\ &= n\pi. \end{aligned} \tag{5.169}$$

To summarize, we have calculated that

$$\int_{\mathbb{R}^2} \omega = \pi \quad \text{and} \quad \int_{\mathbb{R}^2} f_0^* \omega = n\pi. \tag{5.170}$$

Therefore,

$$\deg(f_0) = \deg(f) = n. \tag{5.171}$$

A better way to do the above calculation is in the homework: problem #6 of section 6 of the Supplementary Notes.

Last lecture we showed that if $\deg(f) \neq 0$, then the map f is onto. Applying this to the above example, we find that the algebraic equation

$$z^n + \sum_{i=0}^{n-1} c_i z^i = 0 \tag{5.172}$$

has a solution. This is known as the Fundamental Theorem of Algebra.