Lecture 32

6.4 Tangent Spaces of Manifolds

We generalize our earlier discussion of tangent spaces to tangent spaces of manifolds. First we review our earlier treatment of tangent spaces.

Let $p \in \mathbb{R}^n$. We define

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}. \tag{6.31}$$

Of course, we associate $T_p\mathbb{R}^n \cong \mathbb{R}^n$ by the map $(p, v) \to v$.

If U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^k$, and $f : (U, p) \to (V, q)$ (meaning that f maps $U \to V$ and $p \to p_1$) is a $\mathcal{C}^\infty$ map, then we have the map $df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^k$. Via the identifications $T_p\mathbb{R}^n \cong \mathbb{R}^n$ and $T_p\mathbb{R}^k \cong \mathbb{R}^k$, the map df_p is just the map $Df(p) : \mathbb{R}^n \to \mathbb{R}^k$. Because these two maps can be identified, we can use the chain rule for $\mathcal{C}^\infty$ maps. Specifically, if $f : (U, p) \to (V, q)$ and $g : (V, q) \to (\mathbb{R}^\ell, w)$, then

$$d(g \circ f)_p = (dg)_q \circ (df)_p, \tag{6.32}$$

because $(Dg)(q)(Df(p)) = (Dg \circ f)(p)$.

You might be wondering: Why did we make everything more complicated by using df instead of Df? The answer is because we are going to generalize from Euclidean space to manifolds.

Remember, a set $X \subseteq \mathbb{R}^N$ is an n-dimensional manifold if for every $p \in X$, there exists a neighborhood V of p in $\mathbb{R}^N$, an open set U in $\mathbb{R}^n$, and a diffeomorphism $\phi : U \to V \cap X$. The map $\phi : U \to V \cap X$ is called a parameterization of X at p.

Let us think of ϕ as a map $\phi : U \to \mathbb{R}^N$ with $\text{Im } \phi \subseteq X$.

Claim. *Let $\phi^{-1}(p) = q$. Then the map $(d\phi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N$ is one-to-one.*

Reminder of proof: The map $\phi^{-1} : V \cap X \to U$ is a $\mathcal{C}^\infty$ map. So, shrinking V if necessary, we can assume that this map extends to a map $\psi : V \to U$ such that $\psi = \phi^{-1}$ on $X \cap V$. Then note that for any $u \in U$, we have $\psi(\phi(u)) = \phi^{-1}(\phi(u)) = u$. So, $\psi \circ \phi = \text{id}_U =$ the identity on U.

Using the chain rule, and letting $\phi(q) = p$, we get

$$\begin{aligned} d(\psi \circ \phi)_q &= (d\psi)_o \circ (d\phi)_q \\ &= (d(\text{id}_U))_q. \end{aligned} \tag{6.33}$$

So, $(d\phi)_q$ is injective. □

Today we define for any $p \in X$ the tangent space T_pX, which will be a vector subspace $T_pX \subseteq T_p\mathbb{R}^N$. The tangent space will be like in elementary calculus, that is, a space tangent to some surface.

Let $\phi : U \to V \cap X$ be a parameterization of X, and let $\phi(q) = p$. The above claim tells us that $(d\phi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N$ is injective.

Definition 6.10. We define the *tangent space* of a manifold X to be

$$T_pX = \mathrm{Im}\,(d\phi)_q. \tag{6.34}$$

Because $(d\phi)_q$ is injective, the space T_pX is n-dimensional.

We would like to show that the space T_pX does not depend on the choice of parameterization ϕ. To do so, we will make use of an equivalent definition for the tangent space T_pX.

Last time we showed that given $p \in X \subseteq \mathbb{R}^N$, and $k = N - n$, there exists a neighborhood V of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map $f : V \to \mathbb{R}^k$ mapping $f(p) = 0$ such that $X \cap V = f^{-1}(0)$. Note that $f^{-1}(0) \cap C_f = \phi$ (where here ϕ is the empty set).

We motivate the second definition of the tangent space. Since $p \in f^{-1}(0)$, the point $p \notin C_f$. So, the map $df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$ is surjective. So, the kernel of df_p in $T_p\mathbb{R}^N$ is of dimension $N - k = n$.

Definition 6.11. An alternate definition for the *tangent space* of a manifold is

$$T_pX = \ker df_p. \tag{6.35}$$

Claim. *These two definitions for the tangent space T_pX are equivalent.*

Proof. Let $\phi : U \to V \cap X$ be a parameterization of X at p with $\phi(p) = q$. The function $f : V \to \mathbb{R}^k$ has the property that $f^{-1}(0) = X \cap V$. So, $f \circ \phi \equiv 0$. Applying the chain rule,

$$(df_p) \circ (d\phi_q) = d(0) = 0. \tag{6.36}$$

So, Im $d\phi_q = \ker df_p$. ☐

We can now explain why the tangent space T_pX is independent of the chosen parameterization. We have two definitions for the tangent space. The first does not depend on the choice of ϕ, and the second does not depend on choice of f. Therefore, the tangent space depends on neither.

Lemma 6.12. *Let W be an open subset of $\mathbb{R}^\ell$, and let $g : W \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. Suppose that $g(W) \subseteq X$ and that $g(w) = p$, where $w \in W$. Then $(dg)_W \subseteq T_pX$.*

Proof Hint: We leave the proof as an exercise. As above, we have a map $f : V \to \mathbb{R}^k$ such that $X \cap V = f^{-1}(0)$ and $T_pX = \ker df_p$. Let $W_1 = g^{-1}(V)$, and consider the map $f \circ g : W_1 \to \mathbb{R}^k$. As before, $f \circ g = 0$, so $df_p \circ dg_w = 0$. ☐

Suppose that $X \subseteq \mathbb{R}^N$ is an n-dimensional manifold and $Y \subseteq \mathbb{R}^\ell$ is an m-dimensional manifold. Let $f : X \to Y$ be a $\mathcal{C}^\infty$ map, and let $f(p) = q$.. We want to define a linear map

$$df_p : T_pX \to T_qY. \tag{6.37}$$

Let v be a neighbor hood of p in $\mathbb{R}^N$, and let $g : V \to \mathbb{R}^\ell$ be a map such that $g = f$ on $V \cap X$. By definition $T_pX \subseteq T_p\mathbb{R}^N$, so we have

$$dg_p : T_p\mathbb{R}^N \to T_q\mathbb{R}^k. \tag{6.38}$$

We define the map df_p to be the restriction of dg_p to the tangent space T_pX.

Definition 6.13.

$$df_p = dg_p | T_p X. \tag{6.39}$$

There are two questions about this definition that should have us worried:

1. Is Im $dg_p(T_p X)$ a subset of $T_q Y$?
2. Does this definition depend on the choice of g?

We address these two questions here:

1. Is Im $dg_p(T_p X)$ a subset of $T_q Y$?

   Let U be an open subset of $\mathbb{R}^N$, let $q = f(p)$, and let $\phi : U \to X \cap V$ be a parameterization of X at p. As before, let us think of ϕ as a map $\phi : U \to \mathbb{R}^N$ with $\phi(U) \subseteq X$.

   By definition, $T_p X = \operatorname{Im}(d\phi)_r$, where $\phi(r) = p$. So, given $v \in T_p X$, one can always find $w \in T_r \mathbb{R}^n$ with $v = (d\phi)_r w$.

   Now, is it true that $(dg)_p(v) \in T_q Y$? We have

   $$\begin{aligned}(dg)_p v &= (dg)_p (d\phi)_r (w) \\ &= d(g \circ \phi)_r (w),\end{aligned} \tag{6.40}$$

   and the map $(g \circ \phi)$ is of the form $g \circ \phi : U \to Y$, so

   $$d(g \circ \phi)_r (w) \in T_q Y. \tag{6.41}$$

2. Does the definition depend on the choice of g?

   Consider two such maps $g_1, g_2 : V \to \mathbb{R}^\ell$. The satisfy $g_1 = g_2 = f$ on $X \cap V$. Then, with v, w as above,

   $$(dg_1)_p (v) = d(g_1 \circ \phi)_r (w) \tag{6.42}$$

   $$(dg_2)_p (v) = d(g_2 \circ \phi)_r (w). \tag{6.43}$$

   Since $g_1 = g_2$ on $X \cap V$, we have

   $$g_1 \circ \phi = g_2 \circ \phi = f \circ \phi. \tag{6.44}$$

   Hence,

   $$d(g_1 \circ \phi)_r (w) = d(g_2 \circ \phi)_r (w). \tag{6.45}$$

As an exercise, show that the chain rule also generalizes to manifolds as follows: Suppose that X_1, X_2, X_3 are manifolds with $X_i \subseteq \mathbb{R}^{N_i}$, and let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be $\mathcal{C}^\infty$ maps. Let $f(p) = q$ and $g(q) = r$.

Show the following claim.

Claim.

$$d(g \circ f)_p = (dg_q) \circ (df)_q. \tag{6.46}$$

Proof Hint: Let V_1 be a neighborhood of p in $\mathbb{R}^{N_1}$, and let V_2 be a neighborhood of q in $\mathbb{R}^{N_2}$. Let $\tilde{f} : V_1 \to V_2$ be an extension of f to V_1, and let $\tilde{g} : V_2 \to \mathbb{R}^{N_3}$ be an extension of g to V_2.

The chain rule for f, g follows from the chain rule for $\tilde{f}, \tilde{g}$.

□