Lecture 33

6.5 Differential Forms on Manifolds

Let $U \subseteq \mathbb{R}^n$ be open. By definition, a k-form ω on U is a function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.

We now define the notion of a k-form on a manifold. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold. Then, for $p \in X$, the tangent space $T_pX \subseteq T_p\mathbb{R}^N$.

Definition 6.14. A k-form ω on X is a function on X which assigns to each point $p \in X$ an element $\omega_p \in \Lambda^k((T_pX)^*)$.

Suppose that $f : X \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map, and let $f(p) = a$. Then df_p is of the form

$$df_p : T_pX \to T_a\mathbb{R} \cong \mathbb{R}. \tag{6.47}$$

We can think of $df_p \in (T_pX)^* = \Lambda^1((T_pX)^*)$. So, we get a one-form df on X which maps each $p \in X$ to df_p.

Now, suppose

$$\mu \text{ is a } k\text{-form on } X, \text{ and} \tag{6.48}$$

$$\nu \text{ is an } \ell\text{-form on } X. \tag{6.49}$$

For $p \in X$, we have

$$\mu_p \in \Lambda^k(T_p^*X) \text{ and} \tag{6.50}$$

$$\nu_p \in \Lambda^\ell(T_p^*X). \tag{6.51}$$

Taking the wedge product,

$$\mu_p \wedge \nu_p \in \Lambda^{k+\ell}(T_p^*X). \tag{6.52}$$

The wedge product $\mu \wedge \nu$ is the $(k+\ell)$-form mapping $p \in X$ to $\mu_p \wedge \nu_p$.

Now we consider the pullback operation. Let $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^\ell$ be manifolds, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map. Let $p \in X$ and $a = f(p)$. We have the map

$$df_p : T_pX \to T_aY. \tag{6.53}$$

From this we get the pullback

$$(df_p)^* : \Lambda^k(T_a^*Y) \to \Lambda^k(T_p^*X). \tag{6.54}$$

Let ω be a k-form on Y. Then $f^*\omega$ is defined by

$$(f^*\omega)_p = (df_p)^*\omega_q. \tag{6.55}$$

Let $f : X \to Y$ and $g : Y \to Z$ be $\mathcal{C}^\infty$ maps on manifolds X, Y, Z. Let ω be a k-form. Then

$$(g \circ f)^* \omega = f^*(g^* \omega), \tag{6.56}$$

where $g \circ f : X \to Z$.

So far, the treatment of k-forms for manifolds has been basically the same as our earlier treatment of k-forms. However, the treatment for manifolds becomes more complicated when we study $\mathcal{C}^\infty$ forms.

Let U be an open subset of $\mathbb{R}^n$, and let ω be a k-form on U. We can write

$$\omega = \sum a_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad I = (i_1, \ldots, i_k). \tag{6.57}$$

By definition, we say that $\omega \in \Omega^k(U)$ if each $A_I \in \mathcal{C}^\infty(U)$.

Let V be an open subset of $\mathbb{R}^k$, and let $f : U \to V$ be a $\mathcal{C}^\infty$ map. Let $\omega \in \Omega^k(V)$. Then $f^*\omega \in \Omega^k(U)$. Now, we want to define what we mean by a $\mathcal{C}^\infty$ form on a manifold.

Let $X \subseteq \mathbb{R}^n$ be an n-dimensional manifold, and let $p \in X$. There exists an open set U in $\mathbb{R}^N$, a neighborhood V of p in $\mathbb{R}^N$, and a diffeomorphism $\phi : U \to V \cap X$. The diffeomorphism ϕ is a parameterization of X at p.

We can think of ϕ in the following two ways:

1. as a map of U onto $V \cap X$, or

2. as a map of U onto V, whose image is contained in X.

The second way of thinking about ϕ is actually the map $\iota_X \circ \phi$, where $\iota_X : X \to \mathbb{R}^N$ is the inclusion map. Note that $\iota_X : X \to \mathbb{R}^N$ is $\mathcal{C}^\infty$, because it extends to the identity map $I : \mathbb{R}^N \to \mathbb{R}^N$.

We give two equivalent definitions for $\mathcal{C}^\infty$ k-forms. Let ω be a k-form on X.

Definition 6.15. The k-form ω *is* $\mathcal{C}^\infty$ *at* p if there exists a k-form $\tilde{\omega} \in \Omega^k(V)$ such that $\iota_X^* \tilde{w} = \omega$.

Definition 6.16. The k-form ω *is* $\mathcal{C}^\infty$ *at* p if there exists a diffeomorphism $\phi : U \to V \cap U$ such that $\phi^*\omega \in \Omega^k(U)$.

The first definition depends only on the choice of $\tilde{\omega}$, and the second definition depends only on the choice of ϕ. So, if the definitions are equivalent, then neither definition depends on the choice of $\tilde{\omega}$ or the choice of ϕ.

We show that these two definitions are indeed equivalent.

Claim. *The above two definitions are equivalent.*

Proof. First, we show that (def 6.15) $\implies$ (def 6.16). Let $\omega = \iota_X^* \tilde{\omega}$. Then $\phi^*\omega = (\iota_X \circ \phi)^* \tilde{\omega}$. The map $\iota \circ \phi : U \to V$ is $\mathcal{C}^\infty$, and $\tilde{\omega} \in \Omega^k(v)$, so $\phi^*\omega = (\iota_X \circ \phi)^* \tilde{\omega} \in \Omega^k(U)$.

Second, we show that (def 6.16) $\implies$ (def 6.15). Let $\phi : U \to V \cap U$ be a diffeomorphism. Then $\phi^{-1} : V \cap X \to U$ can be extended to $\psi : V \to U$, where ψ is $\mathcal{C}^\infty$. On $V \cap X$, the map $\phi = \iota_X^* \tilde{\omega}$, where $\tilde{\omega} = \psi^*(\phi^* \omega)$. It is easy to show that $\tilde{\omega}$ is $\mathcal{C}^\infty$. □

Definition 6.17. The k-form ω *is* $\mathcal{C}^\infty$ if ω is $\mathcal{C}^\infty$ at p for every point $p \in X$.

Notation. If ω is $\mathcal{C}^\infty$, then $\omega \in \Omega^k(X)$.

Theorem 6.18. *If $\omega \in \Omega^k(X)$, then there exists a neighborhood W of X in $\mathbb{R}^N$ and a k-form $\tilde{\omega} \in \Omega^k(W)$ such that $\iota_X^* \tilde{\omega} = w$.*

Proof. Let $p \in X$. There exists a neighborhood V_p of p in $\mathbb{R}^N$ and a k-form $\omega^p \in \Omega^k(V_p)$ such that $\iota_X^* \omega^p = \omega$ on $V_p \cap X$.

Let

$$W \subseteq \bigcup_{p \in X} V_p. \tag{6.58}$$

The collection of sets $\{V_p : p \in X\}$ is an open cover of W. Let $\rho_1,\ i = 1, 2, 3, \dots,$ be a partition of unity subordinate to this cover. So, $\rho_i \in \mathcal{C}_0^\infty(W)$ and supp $\rho_i \subset V_p$ for some p. Let

$$\tilde{\omega}_i = \begin{cases} \rho_i \omega^p & \text{on } V_p, \\ 0 & \text{elsewhere.} \end{cases} \tag{6.59}$$

Notice that

$$\begin{aligned} \iota_X^* \tilde{\omega}_i &= \iota_X^* \rho_i \iota_X^* \omega^p \\ &= (\iota_X^* \rho_i) \omega. \end{aligned} \tag{6.60}$$

Take

$$\tilde{\omega} = \sum_{i=1}^{\infty} \tilde{\omega}_i. \tag{6.61}$$

This sum makes sense since we used a partition of unity. From the sum, we can see that $\tilde{w} \in \Omega^k(W)$. Finally,

$$\begin{aligned} \iota_X^* \tilde{w} &= (\iota_X^* \sum \rho_i) \omega \\ &= \omega. \end{aligned} \tag{6.62}$$

□

Theorem 6.19. *Let $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^\ell$ be manifolds, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map. If $\omega \in \Omega^k(X)$, then $f^* \omega \in \Omega^k(Y)$.*

Proof. Take an open set W in $\mathbb{R}^\ell$ such that $W \supset Y$, and take $\tilde{\omega} \in \Omega^k(W)$ such that $\iota_X^* \tilde{\omega} = \omega$. Take any $p \in X$ and $\phi : U \to V$ a parameterization of X at p.

We show that the pullback $\phi^*(f^*\omega)$ is in $\Omega^k(U)$. We can write

$$\begin{aligned} \phi^*(f^*\omega) &= \phi^* f^*(\iota_X^* \tilde{w}) \\ &= (\iota \circ f \circ \phi)^* \tilde{\omega}, \end{aligned} \tag{6.63}$$

where in the last step we used the chain rule.

The form $\tilde{\omega} \in \Omega^k(W)$, where W is open in $\mathbb{R}^\ell$, so $\iota \circ f \circ \phi : U \to W$. The theorem that we proved on Euclidean spaces shows that the r.h.s of Equation 6.63 is in $\Omega^k(U)$. ☐

The student should check the following claim:

Claim. *If $\mu, \nu \in \Omega^k(Y)$, then*

$$f^*(\mu \wedge \nu) = f^*\mu \wedge f^*\nu. \tag{6.64}$$

The differential operation d is an important operator on k-forms on manifolds.

$$d : \Omega^k(X) \to \Omega^{k+1}(X). \tag{6.65}$$

Let $X \subseteq \mathbb{R}^N$ be a manifold, and let $\omega \in \Omega^k(X)$. There exists an open neighborhood W of X in $\mathbb{R}^N$ and a k-form $\tilde{\omega} \in \Omega^k(W)$ such that $\iota_X^* \tilde{\omega} = \omega$.

Definition 6.20. $d\omega = \iota_X^* d\tilde{\omega}$.

Why is this definition well-defined? It seems to depend on the choice of $\tilde{\omega}$.

Take a parameterization $\phi : U \to V \cap X$ of X at p. Then

$$\begin{aligned} \phi^* \iota_X^* d\tilde{\omega} &= (\iota_X \circ \phi)^* d\tilde{\omega} \\ &= d(\iota_X \circ \phi)^* \omega \\ &= d\phi^*(\iota_X^* \tilde{\omega}) \\ &= d\phi^* \omega. \end{aligned} \tag{6.66}$$

So,

$$\phi^* \iota_X^* d\tilde{\omega} = d\phi^* \omega. \tag{6.67}$$

Take the inverse mapping $\phi^{-1} : V \cap X \to U$ and take the pullback $(\phi^{-1})^*$ of each side of Equation 6.67, to obtain

$$\iota_X^* d\tilde{\omega} = (\phi^{-1})^* d\phi^* \omega. \tag{6.68}$$

The r.h.s does not depend on $\tilde{\omega}$, so neither does the l.h.s.

To summarize this lecture, everything we did with k-forms on Euclidean space applies to k-forms on manifolds.