Lecture 36

The first problem on today's homework will be to prove the inverse function theorem for manifolds. Here we state the theorem and provide a sketch of the proof.

Let X, Y be n-dimensional manifolds, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map with $f(p) = p_1$.

Theorem 6.39. *If $df_p : T_pX \to T_{p_1}Y$ is bijective, then f maps a neighborhood V of p diffeomorphically onto a neighborhood V_1 of p_1.*

Sketch of proof: Let $\phi : U \to V$ be a parameterization of X at p, with $\phi(q) = p$. Similarly, let $\phi_1 : U_1 \to V_1$ be a parameterization of Y at p_1, with $\phi_1(q_1) = p_1$.

Show that we can assume that $f : V \to V_1$ (Hint: if not, replace V by $V \cap f^{-1}(V_1)$).

Show that we have a diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V_1 \\ {\scriptstyle\phi}\uparrow & & {\scriptstyle\phi_1}\uparrow \\ U & \xrightarrow{g} & U_1, \end{array} \tag{6.114}$$

which defines g,

$$g = \phi_1^{-1} \circ f \circ \phi, \tag{6.115}$$

$$g(q) = q_1. \tag{6.116}$$

So,

$$(dg)_q = (d\phi_1)_{q_1}^{-1} \circ df_p \circ (d\phi)_q. \tag{6.117}$$

Note that all three of the linear maps on the r.h.s. are bijective, so $(dg)_q$ is a bijection. Use the Inverse Function Theorem for open sets in $\mathbb{R}^n$. $\square$

This ends our explanation of the first homework problem.

Last time we showed the following. Let X, Y be n-dimensional manifolds, and let $f : X \to Y$ be a proper $\mathcal{C}^\infty$ map. We can define a topological invariant $\deg(f)$ such that for every $\omega \in \Omega_c^n(Y)$,

$$\int_X f^*\omega = \deg(f) \int_Y \omega. \tag{6.118}$$

There is a recipe for calculating the degree, which we state in the following theorem. We lead into the theorem with the following lemma.

First, remember that we defined the set C_f of critical points of f by

$$p \in C_f \iff df_p : T_pX \to T_qY \text{ is not surjective}, \tag{6.119}$$

where $q = f(p)$.

Lemma 6.40. *Suppose that $q \in Y - f(C_f)$. Then $f^{-1}(q)$ is a finite set.*

Proof. Take $p \in f^{-1}(q)$. Since $p \notin C_f$, the map df_p is bijective. The Inverse Function Theorem tells us that f maps a neighborhood U_p of p diffeomorphically onto an open neighborhood of q. So, $U_p \cap f^{-1}(q) = p$.

Next, note that $\{U_p : p \in f^{-1}(q)\}$ is an open covering of $f^{-1}(q)$. Since f is proper, $f^{-1}(q)$ is compact, so there exists a finite subcover $U_{p_1}, \dots, U_{p_N}$. Therefore, $f^{-1}(q) = \{p_1, \dots, p_N\}$. □

The following theorem gives a recipe for computing the degree.

Theorem 6.41.

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}, \tag{6.120}$$

where

$$\sigma_{p_i} = \begin{cases} +1 & \textit{if } df_{p_i} : T_{p_i}X \to T_qY \textit{ is orientation preserving,} \\ -1 & \textit{if } df_{p_i} : T_{p_i}X \to T_qY \textit{ is orientation reversing,} \end{cases} \tag{6.121}$$

Proof. The proof is basically the same as the proof in Euclidean space. □

We say that $q \in Y$ is a *regular value of f* if $q \notin f(C_f)$. Do regular values exist? We showed that in the Euclidean case, the set of non-regular values is of measure zero (Sard's Theorem). The following theorem is the analogous theorem for manifolds.

Theorem 6.42. *If $q_0 \in Y$ and W is a neighborhood of q_0 in Y, then $W - f(C_f)$ is non-empty. That is, every neighborhood of q_0 contains a regular value (this is known as the Volume Theorem).*

Proof. We reduce to Sard's Theorem.

The set $f^{-1}(q_0)$ is a compact set, so we can cover $f^{-1}(q_0)$ by open sets $V_i \subset X,\ i = 1, \dots, N$, such that each V_i is diffeomorphic to an open set in $\mathbb{R}^n$.

Let W be a neighborhood of q_0 in Y. We can assume the following:

1. W is diffeomorphic to an open set in $\mathbb{R}^n$,
2. $f^{-1}(W) \subset \bigcup V_i$ (which is Theorem 4.3 in the Supp. Notes),
3. $f(V_i) \subseteq W$ (for, if not, we can replace V_i with $V_i \cap f^{-1}(W)$).

Let U and the sets $U_i,\ i = 1, \dots, N$, be open sets in $\mathbb{R}^n$. Let $\phi : U \to W$ and the maps $\phi_i : U_i \to V_i$ be diffeomorphisms. We have the following diagram:

$$\begin{array}{ccc} V_i & \xrightarrow{f} & W \\ {\scriptstyle \phi_i,\cong}\uparrow & & {\scriptstyle \phi,\cong}\uparrow \\ U_i & \xrightarrow{g_i} & U, \end{array} \tag{6.122}$$

which define the maps g_i,

$$g_i = \phi^{-1} \circ f \circ \phi_i. \tag{6.123}$$

By the chain rule, $x \in C_{g_i} \implies \phi_i(x) \in C_f$, so

$$\phi_i(C_{g_i} = C_f \cap V_i. \tag{6.124}$$

So,

$$\phi(g_i(C_{g_i})) = f(C_f \cap V_i). \tag{6.125}$$

Then,

$$f(C_f) \cap W = \bigcup_i \phi(g_i(C_{g_i})). \tag{6.126}$$

Sard's Theorem tells us that $g_i(C_{g_i})$ is a set of measure zero in U, so

$$U - \bigcup g_i(C_{g_i}) \text{ is non-empty, so} \tag{6.127}$$

$$W - f(C_f) \text{ is also non-empty.} \tag{6.128}$$

In fact, this set is not only non-empty, but is a very, very "full" set. □

Let $f_0, f_1 : X \to Y$ be proper $\mathcal{C}^\infty$ maps. Suppose there exists a proper $\mathcal{C}^\infty$ map $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Then

$$\deg(f_0) = \deg(f_1). \tag{6.129}$$

In other words, the degree is a homotopy. The proof of this is essential the same as before.

6.9 Hopf Theorem

The Hopf Theorem is a nice application of the homotopy invariance of the degree.

Define the n-sphere

$$S^n = \{v \in \mathbb{R}^{n+1} : ||v|| = 1\}. \tag{6.130}$$

Hopf Theorem. *Let n be even. Let $f : S^n \to \mathbb{R}^{n+1}$ be a $\mathcal{C}^\infty$ map. Then, for some $v \in S^n$,*

$$f(v) = \lambda v, \tag{6.131}$$

for some scalar $\lambda \in \mathbb{R}$.

Proof. We prove the contrapositive. Assume that no such v exists, and take $w = f(v)$. Consider $w - \langle v, w \rangle v \equiv w - w_1$. It follows that $w - w_1 \neq 0$.

Define a new map $\tilde{f} : S^n \to S^n$ by

$$\tilde{f}(v) = \frac{f(v) - \langle v, f(x) \rangle}{||f(v) - \langle v, f(x) \rangle||} \tag{6.132}$$

Note that $(w - w_1) \perp v$, so $\tilde{f}(v) \perp v$.

Define a family of functions

$$f_t : S^n \to S^n, \tag{6.133}$$
$$f_t(v) = (\cos t)v + (\sin t)\tilde{w}, \tag{6.134}$$

where $\tilde{w} = \tilde{f}(v)$ has the properties $||\tilde{w}|| = 1$ and $\tilde{w} \perp v$.

We compute the degree of f_t. When $t = 0$, $f_t = \text{id}$, so

$$\deg(f_t) = \deg(f_0) = 1. \tag{6.135}$$

When $t = \pi$, $f_t(v) = -v$. But, if n is even, a map from $S^n \to S^n$ mapping $v \to (-v)$ has degree -1. We have arrived at a contradiction. $\square$