Lecture 38

We begin with a review from last time.

Let X be an oriented manifold, and let $D \subseteq X$ be a smooth domain. Then $\text{Bd}\,(D) = Y$ is an oriented $(n-1)$-dimensional manifold.

We defined integration over D as follows. For $\omega \in \Omega_c^n(X)$ we want to make sense of the integral

$$\int_D \omega. \tag{6.161}$$

We look at some special cases:

Case 1: Let $p \in \text{Int } D$, and let $\phi : U \to V$ be an oriented parameterization of X at p, where $V \subseteq \text{Int } D$. For $\omega \in \Omega_c^n(X)$, we define

$$\int_D \omega = \int_V \omega = \int_U \phi^* \omega = \int_{\mathbb{R}^n} \phi^* \omega. \tag{6.162}$$

This is just our old definition for

$$\int_V \omega. \tag{6.163}$$

Case 2: Let $p \in \text{Bd}\,(D)$, and let $\phi : U \to V$ be an oriented parameterization of D at p. That is, ϕ maps $U \cap \mathbb{H}^n$ onto $V \cap D$. For $\omega \in \Omega_c^n(V)$, we define

$$\int_D \omega = \int_{\mathbb{H}^n} \phi^* \omega. \tag{6.164}$$

We showed last time that this definition does not depend on the choice of parameterization.

General case: For each $p \in \text{Int } D$, let $\phi : U_p \to V_p$ be an oriented parameterization of X at p with $V_p \subseteq \text{Int } D$. For each $p \in \text{Bd}\,(D)$, let $\phi : U_p \to V_p$ be and oriented parameterization of D at p. Let

$$U = \sum_{p \in D} U_p, \tag{6.165}$$

where the set $\mathcal{U} = \{U_p : p \in D\}$ be an an open cover of U. Let $\rho_i,\ i = 1, 2, \dots,$ be a partition of unity subordinate to this cover.

Definition 6.46. For $\omega \in \Omega_c^n(X)$ we define the integral

$$\int_D \omega = \sum_i \int_D \rho_i \omega. \tag{6.166}$$

Claim. *The r.h.s. of this definition is well-defined.*

Proof. Since the ρ_i's are a partition of unity, there exists an N such that

$$\text{supp } \omega \cap \text{supp } \rho_i = \phi, \tag{6.167}$$

for all $i > N$.

Hence, there are only a finite number of non-zero terms in the summand. Moreover, each summand is an integral of one of the two types above (cases 1 and 2), and is therefore well-defined. □

Claim. *The l.h.s. of the definition does not depend on the choice of the partition of unity ρ_i.*

Proof. We proved an analogous assertion about the definition of $\int_X \omega$ a few lectures ago, and the proof of the present claim is exactly the same. □

6.11 Stokes' Theorem

Stokes' Theorem. *For all $\omega \in \Omega_c^{n-1}(X)$,*

$$\int_D d\omega = \int_{\text{Bd}\,(D)} \omega. \tag{6.168}$$

Proof. Let ρ_i, $i = 1, 2 \ldots$, be a partition of unity as defined above. Replacing ω with $\sum \rho_i \omega$, it suffices to prove this for the two special cases below:

Case 1: Let $p \in \text{Int } D$, and let $\phi : U \to V$ be an oriented parameterization of X at p with $V \subseteq \text{Int } D$. If $\omega \in \Omega_c^{n-1}(V)$, then

$$\int_D d\omega = \int_{\mathbb{R}^n} \phi^* d\omega = \int_{\mathbb{R}^n} d\phi^*\omega = 0. \tag{6.169}$$

Case 2: Let $p \in \text{Bd}\,(D)$, and let $\phi : U \to V$ be an oriented parameterization of D at p. Let $U^b = U \cap \text{Bd}\,(\mathbb{H}^n)$, and let $V^b = V \cap \text{Bd}\,(D)$. Define $\psi : \phi|U^b$, so $\psi : U^b \to V^b$ is an oriented parameterization of $\text{Bd}\,(D)$ at p. If $\omega \in \Omega_c^{n-1}(V)$, then

$$\phi^*\omega = \sum f_i(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \tag{6.170}$$

What is $\psi^*\omega$? Let $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the inclusion map mapping $\text{Bd}\,(\mathbb{H}^n) \to \mathbb{R}^n$. The inclusion map ι maps $(x_2, \ldots, x_n) \to (0, x_2, \ldots, x_n)$. Then $\phi \circ \iota = \psi$, so

$$\begin{aligned} \psi^*\omega &= \iota^*\phi^*\omega \\ &= \iota^* \left(\sum_{i=1}^n f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right). \end{aligned} \tag{6.171}$$

But,

$$\iota^* dx_1 = d\iota^* x_1 = 0, \quad \text{since } \iota^* x_1 = 0. \tag{6.172}$$

So,

$$\begin{aligned}\psi^*\omega &= \iota^* f_1 dx_2 \wedge \cdots \wedge dx_n \\ &= f_1(0, x_2, \dots, x_n) dx_2 \wedge \cdots \wedge dx_n.\end{aligned} \tag{6.173}$$

Thus,

$$\int_{\mathrm{Bd}\,(D)} \omega = \int_{\mathbb{R}^{n-1}} \psi^*\omega = \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \dots, x_n) dx_2 \dots dx_n. \tag{6.174}$$

On the other hand,

$$\int_D d\omega = \int_{\mathbb{H}^n} \phi^* d\omega = \int_{\mathbb{H}^n} d\phi^*\omega. \tag{6.175}$$

One should check that

$$\begin{aligned} d\phi^*\omega &= d\left(\sum f_i dx_1 \wedge \cdots \wedge \widehat{x x_i} \wedge \cdots \wedge dx_n\right) \\ &= \left(\sum (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{6.176}$$

So, each summand

$$\int \frac{\partial f_i}{\partial x_i} dx_1 \dots dx_n \tag{6.177}$$

can be integrated by parts, integrating first w.r.t. the ith variable. For $i > 1$, this is the integral

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i &= f_i(x_1, \dots, x_n)\Big|_{x_i=-\infty}^{x_i=\infty} \\ &= 0. \end{aligned} \tag{6.178}$$

For $i = 1$, this is the integral

$$\int_{-\infty}^{\infty} \frac{\partial f_1}{\partial x_1}(x_1, \dots, x_n) dx_1 = f_1(0, x_2, \dots, x_n). \tag{6.179}$$

Thus, the total integral of $\phi^* d\omega$ over $\mathbb{H}^n$ is

$$\int f_1(0, x_2, \dots, x_n) dx_2 \dots dx_n. \tag{6.180}$$

We conclude that

$$\int_D d\omega = \int_{\mathrm{Bd}\,(D)} \omega. \tag{6.181}$$

□

We look at some applications of Stokes' Theorem.

Let D be a smooth domain. Assume that D is compact and oriented, and let $Y = \mathrm{Bd}\,(D)$. Let Z be an oriented n-manifold, and let $f : Y \to Z$ be a $\mathcal{C}^\infty$ map.

Theorem 6.47. *If f extends to a $\mathcal{C}^\infty$ map $F : D \to Z$, then*

$$\deg(f) = 0. \tag{6.182}$$

Corollary 9. *The Brouwer fixed point theorem follows from the above theorem.*