Lecture 6

We begin with a review of some earlier definitions.

Let $\delta > 0$ and $a \in \mathbb{R}^n$.

$$\text{Euclidean ball: } B_\delta(a) = \{x \in \mathbb{R}^n : \| x - a \| < \delta\} \tag{2.66}$$

$$\begin{aligned} \text{Supremum ball: } R_\delta(a) &= \{x \in \mathbb{R}^n : |x - a| < \delta\} \\ &= I_1 \times \cdots \times I_n,\ I_j = (a_j - \delta, a_j + \delta). \end{aligned} \tag{2.67}$$

Note that the supremum ball is actually a rectangle. Clearly, $B_\delta(a) \subseteq R_\delta(a)$. We use the notation $B_\delta = B_\delta(0)$ and $R_\delta = R_\delta(0)$.

Continuing with our review, given U open in $\mathbb{R}^n$, a map $f : U \to \mathbb{R}^k$, and a point $a \in U$, we defined the derivate $Df(a) : \mathbb{R}^n \to \mathbb{R}^k$ which we associated with the matrix

$$Df(a) \sim \left[\frac{\partial f_i}{\partial x_j}(a)\right], \tag{2.68}$$

and we define

$$|Df(a)| = \sup_{i,j} \left|\frac{\partial f_i}{\partial x_j}(a)\right|. \tag{2.69}$$

Lastly, we define $U \subseteq \mathbb{R}^n$ to be *convex* if

$$a, b \in U \implies (1-t)a + tb \in U \text{ for all } 0 \leq t \leq 1. \tag{2.70}$$

Before we state and prove the Inverse Function Theorem, we give the following definition.

Definition 2.13. Let U and V be open sets in $\mathbb{R}^n$ and $f : U \to V$ a $\mathcal{C}^r$ map. The map f is is a $\mathcal{C}^r$ *diffeomorphism* if it is bijective and $f^{-1} : V \to U$ is also $\mathcal{C}^r$.

Inverse Function Theorem. *Let U be an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ a $\mathcal{C}^r$ map, and $a \in U$. If $Df(a) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, then there exists a neighborhood U_1 of a in U and a neighborhood V of $f(a)$ in $\mathbb{R}^n$ such that $F|U_1$ is a $\mathcal{C}^r$ diffeomorphism of U_1 at V.*

Proof. To prove this we need some elementary multi-variable calculus results, which we provide with the following lemmas.

Lemma 2.14. *Let U be open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^k$ be a $\mathcal{C}^1$ mapping. Also assume that U is convex. Suppose that $|Df(a)| \leq c$ for all $A \in U$. Then, for all $x, y \in U$,*

$$|f(x) - f(y)| \leq nc|x - y|. \tag{2.71}$$

Proof. Consider any $x, y \in U$. The Mean Value Theorem says that for every i there exists a point c on the line joining x to y such that

$$f_i(x) - f_i(y) = \sum_j \frac{\partial f_i}{\partial x_j}(d)(x_j - y_j). \tag{2.72}$$

It follows that

$$\begin{aligned} |f_i(x) - f_i(y)| &\leq \sum_j \left| \frac{\partial f_i}{\partial x_j}(d) \right| |x_j - y_j| \\ &\leq \sum_j c|x_j - y_j| \\ &\leq nc|x - y| \end{aligned} \tag{2.73}$$

This is true for each i, so $|f(x) - f(y)| \leq nc|x - y|$ □

Lemma 2.15. *Let U be open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^1$ map. Suppose f takes a minimum value at some point $b \in U$. Then*

$$\frac{\partial f}{\partial x_i}(b) = 0, \ i = 1, \dots, n. \tag{2.74}$$

Proof. We reduce to the one-variable result. Let $b = (b_1, \dots, b_n)$ and let $\phi(t) = f(b_1, \dots, b_{i-1}, t, b_{i+1}, \dots, b_n)$, which is $\mathcal{C}^1$ near b_1 and has a minimum at b_i. We know from one-variable calculus that this implies that $\frac{\partial \phi}{\partial t}(b_i) = 0$. □

In our proof of the Inverse Function Theorem, we want to show that f is locally a diffeomorphism at a. We will make the following simplifying assumptions:

$$a = 0, \quad f(a) = 0, \quad Df(0) = I \text{ (identity)}. \tag{2.75}$$

Then, we define a map $g : U \to \mathbb{R}^n$ by $g(x) = x - f(x)$, so that we obtain the further simplification

$$Dg(0) = Df(0) - I = 0. \tag{2.76}$$

Lemma 2.16. *Given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in R_\delta$,*

$$|g(x - g(y)| < \epsilon|x - y|. \tag{2.77}$$

Proof. The result that $Dg(0) = 0$ implies that there exists $\delta > 0$ such that for any $x \in R_\delta$, $|Dg(x)| \leq \epsilon/n$. Applying the first lemma, the proof is complete. □

Now, remember that $g(x) = x - f(x)$. Take any $x, y \in R_\delta$. Then

$$\begin{aligned} x - y &= x - f(x) + f(x) - f(y) + f(y) - y \\ &= g(x) - g(y) + f(x) - f(y). \end{aligned} \tag{2.78}$$

Using the Triangle Inequality we obtain

$$|x-y| \leq |g(x)-g(y)| + |f(x)-f(y)| \tag{2.79}$$

Using the previous lemma, we find that

$$(1-\epsilon)|x-y| \leq |f(x)-f(y)|. \tag{2.80}$$

We choose δ such that $\epsilon > 1/2$, so that

$$|x-y| \leq 2|f(x)-f(y)|. \tag{2.81}$$

This proves that $f : R_\delta \to \mathbb{R}^n$ is one-to-one.

We also want to prove that f is onto. We have $Df(0) = I$, so $\det(\frac{\partial f_i}{\partial x_j}(0)) = 1$. We can choose δ such that for any $x \in R_\delta$,

$$\det\left(\frac{\partial f_i}{\partial x_j}(x)\right) > \frac{1}{2}. \tag{2.82}$$

Lemma 2.17. *If $y \in B_{\delta/4}$, than there exists a point $c \in R_\delta$ such that $f(c) = y$.*

Proof. Let $h : \bar{R}_\delta \to \mathbb{R}$ be a map defined by $h(x) = \| f(x) - y \|^2$. The domain $\bar{R}_\delta$ is compact, so h has a minimum at some point $c \in \bar{R}_\delta$.

Claim. *The point c is an interior point. That is, $c \in R_\delta$.*

Proof. For any $x \in \bar{R}_\delta$, $|x| = \delta$ implies that $|f(x) - f(0)| = |f(x)| \geq \delta/2$

$$\begin{aligned} &\implies \| f(x) \| \geq \frac{\delta}{2} \\ &\implies \| f(x) - y \| \geq \frac{\delta}{4}, \text{ when } x \in \text{Bd } R_\delta. \\ &\implies h(x) \geq \left(\frac{\delta}{4}\right)^2. \end{aligned} \tag{2.83}$$

At the origin, $h(0) = \| f(0) - y \|^2 = \| y \|^2 < (\delta/4)^2$, since $y \in B_{\delta/4}$. So, $h(0) \leq h$ on Bd R_δ, which means that the minimum point c of h is in R_δ. This ends the proof of the claim. □

Now that we know that the minimum point c occurs in the interior, we can apply the second lemma to h to obtain

$$\frac{\partial h}{\partial x_j}(c) = 0, \quad j = 1, \ldots, n. \tag{2.84}$$

From the definition of h,

$$h(x) = \sum_{i=1}^{n} (f_i(c) - y_i) \frac{\partial f_i}{\partial x_j}(c) = 0, \; i = 1, \ldots, n, \tag{2.85}$$

so

$$\frac{\partial h}{\partial x_j}(c) = 2\sum_{i-1}^{n}(f_i(c) - y_i)\frac{\partial f_i}{\partial x_j}(c) = 0,\ i = 1, \ldots, n. \tag{2.86}$$

Note that

$$\det\left[\frac{\partial f_i}{\partial x_j}(c)\right] \neq 0, \tag{2.87}$$

so, by Cramer's Rule,

$$f_i(c) - y_i = 0,\ i = 1, \ldots, n. \tag{2.88}$$

□

Let $U_1 = R_\delta \sim f^{-1}(B_{\delta/4})$, where we have chosen $V = B_{\delta/4}$. We have shown that f is a bijective map.

Claim. *The map $f^{-1} : V \to U_1$ is continuous.*

Proof. Let $a, b \in V$, and define $x = f^{-1}(a)$ and $y = f^{-1}(b)$. Then $a = f(x)$ and $b = f(y)$.

$$|a - b| = |f(x) - f(y)| \geq \frac{\partial |x - y|}{\partial 2}, \tag{2.89}$$

so

$$|a - b| \geq \frac{1}{2}|f^{-1}(a) - f^{-1}(b)|. \tag{2.90}$$

This shows that f^{-1} is continuous on $V = B_{\delta/4}$. □

As a last item for today's lecture, we show the following:

Claim. *The map f^{-1} is differentiable at 0, and $Df^{-1}(0) = I$.*

Proof. Let $k \in \mathbb{R}^n - \{0\}$ and choose $k \doteq 0$. We are trying to show that

$$\frac{f^{-1}(0 + k) - f^{-1}(0) - Df^{-1}(0)k}{|k|} \to 0 \text{ as } k \to 0. \tag{2.91}$$

We simplify

$$\frac{f^{-1}(0 + k) - f^{-1}(0) - Df^{-1}(0)k}{|k|} = \frac{f^{-1}(k) - k}{|k|}. \tag{2.92}$$

Define $h = f^{-1}(k)$ so that $k = f(h)$ and $|k| \leq 2|h|$. To show that

$$\frac{f^{-1}(k) - k}{|k|} \to 0 \text{ as } k \to 0, \tag{2.93}$$

it suffices to show that

$$\frac{f^{-1}(k) - k}{|h|} \to 0 \text{ as } h \to 0. \tag{2.94}$$

That is, it suffices to show that

$$\frac{h - f(h)}{|h|} \to 0 \text{ as } h \to 0. \tag{2.95}$$

But this is equal to

$$-\frac{f(h) - f(0) - Df(0)h}{|h|}, \tag{2.96}$$

which goes to zero as $h \to 0$ because f is differentiable at zero. □

The proof of the Inverse Function Theorem continues in the next lecture.