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18.102 Introduction to Functional Analysis  
Spring 2009

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## PROBLEMS 6: DUE 11AM TUESDAY, 31 MAR

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpful. An example being the old hint for Problem 6.2!

*Problem 6.1* Let  $H$  be a separable Hilbert space. Show that  $K \subset H$  is compact if and only if it is closed, bounded and has the property that any sequence in  $K$  which is weakly convergent sequence in  $H$  is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

*Problem 6.2* Show that, in a separable Hilbert space, a weakly convergent sequence  $\{v_n\}$ , is (strongly) convergent if and only if the weak limit,  $v$  satisfies

$$(12.19) \quad \|v\|_H = \lim_{n \rightarrow \infty} \|v_n\|_H.$$

Hint:- To show that this condition is sufficient, expand

$$(12.20) \quad (v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2.$$

*Problem 6.3* Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any  $\epsilon > 0$  there exists a linear subspace  $D_N \subset H$  of finite dimension such that

$$(12.21) \quad d(K, D_N) = \sup_{u \in K} \inf_{v \in D_N} \{d(u, v)\} \leq \epsilon.$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in  $K$  is strongly convergent, use the convexity result from class to define the sequence  $\{v'_n\}$  in  $D_N$  where  $v'_n$  is the closest point in  $D_N$  to  $v_n$ . Show that  $v'_n$  is weakly, hence strongly, convergent and hence deduce that  $\{v_n\}$  is Cauchy.

*Problem 6.4* Suppose that  $A : H \rightarrow H$  is a bounded linear operator with the property that  $A(H) \subset H$  is finite dimensional. Show that if  $v_n$  is weakly convergent in  $H$  then  $Av_n$  is strongly convergent in  $H$ .

*Problem 6.5* Suppose that  $H_1$  and  $H_2$  are two different Hilbert spaces and  $A : H_1 \rightarrow H_2$  is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint)  $A^* : H_2 \rightarrow H_1$  with the property

$$(12.22) \quad \langle Au_1, u_2 \rangle_{H_2} = \langle u_1, A^*u_2 \rangle_{H_1} \quad \forall u_1 \in H_1, u_2 \in H_2.$$