18.102 Introduction to Functional Analysis
Spring 2009

## Lecture 6. Tuesday, Feb 24

By now the structure of the proofs should be getting somewhat routine – but I will go on to the point that I hope it all becomes clear!

So, recall the definitions of a Lebesgue integrable function on the line (forming the linear space $\mathcal{L}^1(\mathbb{R})$) and of a set of measure zero $E \subset \mathbb{R}$.

The first thing we want to show is that the putative norm on $L^1$ does make sense.

**Proposition 9.** *If $f \in \mathcal{L}^1(\mathbb{R})$ then $|f| \in \mathcal{L}^1(\mathbb{R})$ and if $f_n$ is an absolutely summable series of step functions converging to $f$ almost everywhere then*

$$\int |f| = \lim_{N\to\infty} \int |\sum_{k=1}^{N} f_k|. \tag{6.1}$$

So in some sense the definition of the Lebesgue integral 'involves no cancellations'. There are extensions of the integral, we may even see one, which do exploit cancellations.

*Proof.* By definition if $f \in \mathcal{L}^1(\mathbb{R})$ then it is the limit, on the set of absolute convergence, of a summable series of step functions, $\{f_n\}$. We need to make such a series for $|f|$. The idea in this case is the 'obvious' one. We know that

$$\sum_{j=1}^{n} f_j(x) \to f(x) \text{ if } \sum_j |f_j(x)| < \infty. \tag{6.2}$$

So, set

$$g_1(x) = |f_1(x)|,\ g_k(x) = |\sum_{j=1}^{k} f_j(x)| - |\sum_{j=1}^{k-1} f_j(x)| \ \forall\ x \in \mathbb{R}. \tag{6.3}$$

Then, for sure,

$$\sum_{k=1}^{N} g_k(x) = |\sum_{j=1}^{N} f_j(x)| \to |f(x)| \text{ if } \sum_j |f_j(x)| < \infty. \tag{6.4}$$

So, what we need to check, for a start, is that $\{g_j\}$ is an absolutely summable series of step functions.

The triangle inequality in the form $||v| - |w|| \le |v - w|$ shows that, for $k > 1$,

$$|g_k(x)| = ||\sum_{j=1}^{k} f_j(x)| - |\sum_{j=1}^{k-1} f_j(x)|| \le |f_k(x)|. \tag{6.5}$$

Thus

$$\sum_k \int |g_k| \le \sum_k \int |f_k| < \infty \tag{6.6}$$

so the $g_k$'s do indeed form an absolutely summable series. From its construction we know that

$$\sum_{k=1}^{N} g_k(x) = |\sum_{j=1}^{N} f_j(x)| \to |f(x)| \text{ if } \sum_n |f_n(x)| < \infty. \tag{6.7}$$

So, this is what we want except that the set on which $\sum_k |g_k(x)| < \infty$ may be larger than the set for which we have convergence here. Now, in the notes there is a result to handle this, but we can simply make the series converge less rapidly by adding a 'pointless' subseries. Namely replace $g_k$ by

$$h_n(x) = \begin{cases} g_k(x) & \text{if } n = 3k-2 \\ f_k(x) & \text{if } n = 3k-1 \\ -f_k(x) & \text{if } n = 3k. \end{cases} \tag{6.8}$$

This series converges absolutely if and only if *both* the $|g_k(x)|$ and $|f_k(x)|$ series converge – the convergence of the latter implies the convergence of the former so

$$\sum_n |h_n(x)| < \infty \Longleftrightarrow \sum_k |f_k(x)|. \tag{6.9}$$

On the other hand when this holds,

$$\sum_n h_n(x) = |f(x)| \tag{6.10}$$

since each partial sum is either a sum for $g_k$, or this with $f_n(x)$ added. Since $f_n(x) \to 0$, (6.10) holds whenever the series converges absolutely, so indeed $|f| \in \mathcal{L}^1(\mathbb{R})$.

Then (6.1) follows. □

Now, the next thing we want to know is when 'norm' vanishes. That is, when does $\int |f| = 0$? One way is fairly easy, I think I actually skated over this a bit in the lecture, so let me write it out carefully here. The result we are after is:-

**Proposition 10.** *For an integrable function $f \in \mathcal{L}^1(\mathbb{R})$, the vanishing of $\int |f|$ implies that $f$ is a null function in the sense that*

$$f(x) = 0 \ \forall \ x \in \mathbb{R} \setminus E \text{ where } E \text{ is of measure zero.} \tag{6.11}$$

*Conversely, if* (6.11) *holds then $f \in \mathcal{L}^1(\mathbb{R})$ and $\int |f| = 0$.*

*Proof.* The main part of this is the first part, that the vanishing of $\int |f|$ implies that $f$ is null. This I will prove using the next Proposition. The converse is the easier direction.

Namely, if $f$ is null in the sense of (6.11) then, by the definition of a set of measure zero, there exists an absolutely summable series of step functions, $f_n$, such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\}. \tag{6.12}$$

Note that it is possible that the absolute series here diverges on a larger set than $E$. Still, if we consider the alternating series

$$g_n(x) = \begin{cases} f_k(x) & \text{if } n = 2k-1 \\ -f_k(x) & \text{if } n = 2k \end{cases} \tag{6.13}$$

then

$$\sum_n g_n(x) = 0 \text{ whenever} \tag{6.14}$$

since the latter condition is equivalent to $\sum_n |f_n(x)| < \infty$. So in fact

$$\sum_n |g_n(x)| < \infty \Longrightarrow f(x) = \sum_n g_n(x) = 0 \tag{6.15}$$

because of (6.12). Thus the null function $f \in \mathcal{L}^1(\mathbb{R})$, and so is $|f|$ and from (6.15)

$$\int |f| = \sum_k \int g_k = \lim_{k\to\infty} \int f_k = 0 \tag{6.16}$$

where the last statement follows from the absolute summability. □

For the converse argument we will use the following result, which is also closely related to the completeness of $L^1(\mathbb{R})$.

**Proposition 11.** *If $f_n \in \mathcal{L}^1(\mathbb{R})$ is an absolutely summable series, in the sense that $\sum_n \int |f_n| < \infty$ then*

$$E = \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\} \text{ has measure zero} \tag{6.17}$$

*and if $f : \mathbb{R} \longrightarrow \mathbb{C}$ and*

$$f(x) = \sum_n f_n(x) \ \forall \ x \in \mathbb{R} \setminus E \tag{6.18}$$

*then $f \in \mathcal{L}^1(\mathbb{R})$ and*

$$\int f = \sum_n \int f_n. \tag{6.19}$$

So this basically says we can replace 'step function' by 'integrable function' in the definition and get the same result. Of course this makes no sense without the original definition.

*Proof.* The proof is very like the proof of completeness of the 'completion' of a normed space that was in Problems 2, here it is a little more concrete.

Thus, by assumption each $f_n \in \mathcal{L}^1(\mathbb{R})$, so there exists an absolutely summable series of step functions $f_{n,j}$, so $\sum_j \int |f_{n,j}| < \infty$ and

$$\sum_j |f_{n,j}(x)| < \infty \Longrightarrow f_n(x) = \sum_j f_{n,j}(x). \tag{6.20}$$

We can expect $f(x)$ to be given by the sum of the $f_{n,j}(x)$ over both $n$ and $j$, but in general, this double series is not absolutely summable. However we can make it so. Simply choose $N_n$ for each $n$ so that

$$\sum_{j>N_n} \int |f_{n,j}| < 2^{-n}. \tag{6.21}$$

This is possible by the assumed absolute summability – so the tail of the series is small. Having done this, we replace the series $f_{n,j}$ by

$$f'_{n,1} = \sum_{j\le N_n} f_{n,j}(x),\ f'_{n,j}(x) = f_{n,N_n+k-1}(x) \ \forall \ j \ge 2. \tag{6.22}$$

This still converges to $f_n$ on the same set as in (6.20). So in fact we can simply replace $f_{n,j}$ by $f'_{n,j}$ and we have in addition the estimate

$$\sum_j \int |f'_{n,j}| \le \int |f_n| + 2^{-n+1} \ \forall \ n. \tag{6.23}$$

This follows from the triangle inequality since, using (6.21),

$$\int |f'_{n,1} + \sum_{k=1}^{N} f'_{n,j}| \ge \int |f'_{n,1}| - \sum_{j\ge 2} \int |f'_{n,j}| \ge \int |f'_{n,1}| - 2^{-n} \tag{6.24}$$

and the left side converges to $\int |f_n|$ by (6.1) as $N \to \infty$. Using (6.21) again gives (6.23).

So, now dropping the primes from the notation and using the new series as $f_{n,j}$ we can set

$$g_k(x) = \sum_{n+j=k} f_{n,j}. \tag{6.25}$$

This gives a new series of step functions which is absolutely summable since

$$\sum_{k=1}^{N} \int |g_k| \le \sum_{n,j} \int |f_{n,j}| \le \sum_n (\int |f_n| + 2^{-n+1}) < \infty. \tag{6.26}$$

Now, using rearrangement of absolutely convergent series we see that

$$\sum_{n,j} |f_{n,j}(x)| < \infty \Longrightarrow f(x) = \sum_k |g_k(x)| = \sum_n \sum_j f_{n,j}(x). \tag{6.27}$$

From a result last week, we know that the set on the left here is of the form $\mathbb{R} \setminus E$ where $E$ is of measure zero; $E$ is the union of the sets $E_n$ on which $\sum_j |f_{n,j}(x)| = \infty$. So, take another absolutel summable series of step functions $h_k$ which diverges on $E$ (at least) and insert $h_k$ and $-h_k$ between successive $g_k$'s as before. This new series still coverges to $f$ by (6.27) and shows that $f \in \mathcal{L}^1(\mathbb{R})$ as well as (6.17). The final result (6.19) also follows by rearranging the double series for the integral (which is also absolutely convergent). □

Now for the moment we only need the weakest part, (6.17), of this. That is for any absolutely summable series of integrable functions the absolute pointwise series converges off a set of measure zero – can only diverge on a set of measure zero. It is rather shocking but this allows us to prove the rest of (10)! Namely, suppose $f \in \mathcal{L}^1(\mathbb{R})$ and $\int |f| = 0$. Then Proposition 11 applies to the series with each term being $|f|$. This is absolutely summable since all the integrals are zero. So it must converge pointwise except on a set of measure zero. Clearly it diverges whenever $f(x) \neq 0$, so

$$\int |f| = o \Longrightarrow \{x; f(x) \neq 0\} \text{ has measure zero} \tag{6.28}$$

which is what we wanted to show.

Finally this allows us to define the standard Lebesgue space

$$L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}, \ \mathcal{N} = \{\text{null functions}\} \tag{6.29}$$

and to check that $\int |f|$ is indeed a norm on this space.

## Problem set 3, Due 11AM Tuesday 3 Mar.

This problem set is also intended to be a guide to what will be on the in-class test on March 5. In particular I will ask you to prove some of the properties of the Lebesgue integral, as below, plus one more abstract proof. Recall that equality a.e (almost everywhere) means equality on the complement of a set of measure zero.

*Problem 3.1* If $f$ and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

(1) If $f(x) \ge 0$ a.e. then $\int f \ge 0$.
(2) If $f(x) \le g(x)$ a.e. then $\int f \le \int g$.
(3) If $f$ is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \le \int |f|$.
(4) For a general complex-valued Lebesgue integrable function

$$|\int f| \le \int |f|. \tag{6.30}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0, 2\pi)$ so that $e^{i\theta} \int f = \int (e^{i\theta} f) \ge 0$. Then apply the preceeding estimate to $g = e^{i\theta} f$.

(5) Show that the integral is a continuous linear functional

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{6.31}$$

*Problem 3.2* If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or $(a, \infty)$, we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C}, \ \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{6.32}$$

The integral of $f$ on $I$ is then defined to be

$$\int_I f = \int \tilde{f}. \tag{6.33}$$

(1) Show that the space of such integrable functions on $I$ is linear, denote it $\mathcal{L}^1(I)$.
(2) Show that is $f$ is integrable on $I$ then so is $|f|$.
(3) Show that if $f$ is integrable on $I$ and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.
(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C}, \ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{6.34}$$

is integrable on $I$.

(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to $I$'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{6.35}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

*Problem 3.3* Really continuing the previous one.

(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of $f$ to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.
(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{6.36}$$

is continuous.
(3) Prove that the function $x^{-1}\cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

*Problem 3.4* [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x - t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{6.37}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.
(2) Show that

$$\lim_{t \to 0} \int |f_t - f| = 0. \tag{6.38}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!
(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{6.39}$$

is continuous.

*Problem 3.5* In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

*Problem 3.6*

(1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \le \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{6.40}$$

(2) Suppose now that $G \in \mathcal{C}([0, 1] \times [0, 1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0, 1])$. Now, using the first part show that if $f \in L^1([0, 1])$ then

$$F(x) = \int_{[0,1]} G(x, \cdot) f(\cdot) \in \mathbb{C} \tag{6.41}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0,1]$.

(3) Show that for each $f \in L^1([0,1])$, $F$ is a continuous function on $[0,1]$.

(4) Show that

$$L^1([0,1]) \ni f \longmapsto F \in \mathcal{C}([0,1]) \tag{6.42}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0,1]$.

## Solutions to Problem set 2

I was originally going to make this problem set longer, since there is a missing Tuesday. However, I would prefer you to concentrate on getting all four of these questions really right!

*Problem 2.1* Finish the proof of the completeness of the space $B$ constructed in lecture on February 10. The description of that construction can be found in the notes to Lecture 3 as well as an indication of one way to proceed.

*Solution.* The proof could be shorter than this, I have tried to be fairly complete.

To recap. We start of with a normed space $V$. From this normed space we construct the new linear space $\tilde{V}$ with points the absolutely summable series in $V$. Then we consider the subspace $S \subset \tilde{V}$ of those absolutely summable series which converge to 0 in $V$. We are interested in the quotient space

$$B = \tilde{V}/S. \tag{6.43}$$

What we know already is that this is a normed space where the norm of $b = \{v_n\}+S$ – where $\{v_n\}$ is an absolutely summable series in $V$ is

$$\|b\|_B = \lim_{N\to\infty} \|\sum_{n=1}^{N} v_n\|_V. \tag{6.44}$$

This is independent of which series is used to represent $b$ – i.e. is the same if an element of $S$ is added to the series.

Now, what is an absolutely summable series in $B$? It is a sequence $\{b_n\}$, thought of a series, with the property that

$$\sum_n \|b_n\|_B < \infty. \tag{6.45}$$

We have to show that it converges in $B$. The first task is to guess what the limit should be. The idea is that it should be a series which adds up to 'the sum of the $b_n$'s'. Each $b_n$ is represented by an absolutely summable series $v_k^{(n)}$ in $V$. So, we can just look for the usual diagonal sum of the double series and set

$$w_j = \sum_{n+k=j} v_k^{(n)}. \tag{6.46}$$

The problem is that this will not in generall be absolutely summable as a series in $V$. What we want is the estimate

$$\sum_j \|w_j\| = \sum_j \| \sum_{j=n+k} v_k^{(n)}\| < \infty. \tag{6.47}$$

The only way we can really estimate this is to use the triangle inequality and conclude that

$$\sum_{j=1}^{\infty} \|w_j\| \le \sum_{k,n} \|v_k^{(n)}\|_V. \tag{6.48}$$

Each of the sums over $k$ on the right is finite, but we do not know that the sum over $k$ is then finite. This is where the first suggestion comes in:-

We can *choose* the absolutely summable series $v_k^{(n)}$ representing $b_n$ such that

$$\sum_k \|v_k^{(n)}\| \le \|b_n\|_B + 2^{-n}. \tag{6.49}$$

Suppose an initial choice of absolutely summable series representing $b_n$ is $u_k$, so $\|b_n\| = \lim_{N\to\infty} \|\sum\limits_{k=1}^{N} u_k\|$ and $\sum\limits_k \|u_k\|_V < \infty$. Choosing $M$ large it follows that

$$\sum_{k>M} \|u_k\|_V \le 2^{-n-1}. \tag{6.50}$$

With this choice of $M$ set $v_1^{(n)} = \sum\limits_{k=1}^{M} u_k$ and $v_k^{(n)} = u_{M+k-1}$ for all $k \ge 2$. This does still represent $b_n$ since the difference of the sums,

$$\sum_{k=1}^{N} v_k^{(n)} - \sum_{k=1}^{N} u_k = -\sum_{k=N}^{N+M-1} u_k \tag{6.51}$$

for all $N$. The sum on the right tends to 0 in $V$ (since it is a fixed number of terms). Moreover, because of (6.50),

$$\sum_k \|v_k^{(n)}\|_V = \|\sum_{j=1}^{M} u_j\|_V + \sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2\sum_{k>M} \|u_k\| \le \|\sum_{j=1}^{N} u_j\| + 2^{-n} \tag{6.52}$$

for all $N$. Passing to the limit as $N \to \infty$ gives (6.49).

Once we have chosen these 'nice' representatives of each of the $b_n$'s if we define the $w_j$'s by (6.46) then (6.47) means that

$$\sum_j \|w_j\|_V \le \sum_n \|b_n\|_B + \sum_n 2^{-n} < \infty \tag{6.53}$$

because the series $b_n$ is absolutely summable. Thus $\{w_j\}$ defines an element of $\tilde{V}$ and hence $b \in B$.

Finally then we want to show that $\sum\limits_n b_n = b$ in $B$. This just means that we need to show

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|_B = 0. \tag{6.54}$$

The norm here is itself a limit – $b - \sum\limits_{n=1}^{N} b_n$ is represented by the summable series with $n$th term

$$w_k - \sum_{n=1}^{N} v_k^{(n)} \tag{6.55}$$

and the norm is then

$$\lim_{p\to\infty} \|\sum_{k=1}^{p} (w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V. \tag{6.56}$$

Then we need to understand what happens as $N \to \infty$! Now, $w_k$ is the diagonal sum of the $v_j^{(n)}$'s so sum over $k$ gives the difference of the sum of the $v_j^{(n)}$ over the first $p$ anti-diagonals minus the sum over a square with height $N$ (in $n$) and width

$p$. So, using the triangle inequality the norm of the difference can be estimated by the sum of the norms of all the 'missing terms' and then some so

$$\|\sum_{k=1}^{p}(w_k - \sum_{n=1}^{N} v_k^{(n)})\|_V \le \sum_{l+m\ge L} \|v_l^{(m)}\|_V \tag{6.57}$$

where $L = \min(p, N)$. This sum is finite and letting $p \to \infty$ is replaced by the sum over $l + m \ge N$. Then letting $N \to \infty$ it tends to zero by the absolute (double) summability. Thus

$$\lim_{N\to\infty} \|b - \sum_{n=1}^{N} b_n\|B = 0 \tag{6.58}$$

which is the statelent we wanted, that $\sum_n b_n = b$. □

*Problem 2.2* Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0, 1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3, 2/3)$. This leave $C_1 = [0, 1/3) \cup [2/3, 1)$. Then remove the central interval from each of the remaining two intervals to get $C_2 = [0, 1/9) \cup [2/9, 1/3) \cup [2/3, 7/9) \cup [8/9, 1)$. Carry on in this way to define successive sets $C_k \subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions $f_k$ where $f_k(x) = 1$ on $C_k$ and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x \in [0, 1)$ does $\sum_k |f_k(x)|$ converge?
(3) Describe a function on $[0, 1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function $g$ which is equal to one on the union of all the subintervals of $[0, 1)$ which are *removed* in the construction and zero elsewhere. Show that $g$ is Lebesgue integrable and compute its integral.

*Solution.* (1) The total length of the intervals is being reduced by a factor of 1/3 each time. Thus $l(C_k) = \frac{2^k}{3^k}$. Thus the integral of $f$, which is non-negative, is actually

$$\int f_k = \frac{2^k}{3^k} \Longrightarrow \sum_k \int |f_k| = \sum_{k=1}^{\infty} \frac{2^k}{3^k} = 2 \tag{6.59}$$

Thus the series is absolutely summable.

(2) Since the $C_k$ are decreasing, $C_k \supset C_{k+1}$, only if

$$x \in E = \bigcap_k C_k \tag{6.60}$$

does the series $\sum_k |f_k(x)|$ diverge (to $+\infty$) otherwise it converges.

(3) The function defined as the sum of the series where it converges and zero otherwise

$$f(x) = \begin{cases} \sum_k f_k(x) & x \in \mathbb{R} \setminus E \\ 0 & x \in E \end{cases} \tag{6.61}$$

is integrable by definition. Its integral is by definition

$$\int f = \sum_k \int f_k = 2 \tag{6.62}$$

from the discussion above.

(4) The function $f$ is not Riemann integrable since it is not bounded – and this is part of the definition. In particular for $x \in C_k \setminus C_{k+1}$, which is not an empty set, $f(x) = k$.

(5) The set $F$, which is the union of the intervals removed is $[0, 1) \setminus E$. Taking step functions equal to 1 on each of the intervals removed gives an absolutely summable series, since they are non-negative and the $k$th one has integral $1/3 \times (2/3)^{k-1}$ for $k = 1, \dots$. This series converges to $g$ on $F$ so $g$ is Lebesgue integrable and hence

$$\int g = 1. \tag{6.63}$$

□

*Problem 2.3* The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1, b_1) \times [a_2, b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1 - a_1) \times (b_2 - a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1, b_1)$ by $[a_1, c_1) \cup [c_1, b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.

(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.

(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.

(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.

(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

*Solution.* (1) For subdivision of one rectangle this is clear enough. Namely we either divide the first side into two or the second side in two at an intermediate point $c$. After subdivision the area of the two rectanges is either

$$\begin{gathered}(c - a_1)(b_2 - a_2) + (b_1 - c)(b_2 - a_2) = (b_1 - c_1)(b_2 - a_2) \text{ or} \\ (b_1 - a_1)(c - a_2) + (b_1 - a_1)(b_2 - c) = (b_1 - c_1)(b_2 - a_2).\end{gathered} \tag{6.64}$$

this shows by induction that the sum of the areas of any the rectangles made by repeated subdivision is always the same as the original.

(2) If a finite collection of disjoint rectangles has union a rectangle, say $[a_1, b_2) \times [a-2, b_2)$ then the same is true after any subdivision of any of the rectangles, by the previous result. Moreover after such subdivision the sum of the areas is always the same. Look at all the points $C_1 \subset [a_1, b_1)$ which occur as an endpoint of the first interval of one of the rectangles. Similarly let $C_2$ be the corresponding set of end-points of the second intervals of the rectangles. Now divide each of the rectangles repeatedly using the finite number of points in $C_1$ and the finite number of points in $C_2$. The total area remains the same and now the rectangles covering $[a_1, b_1) \times [A_2, b_2)$ are precisely the $A_i \times B_j$ where the $A_i$ are a set of disjoint intervals covering $[a_1, b_1)$ and the $B_j$ are a similar set covering $[a_2, b_2)$. Applying the one-dimensional result from class we see that the sum of the areas of the rectangles with first interval $A_i$ is the product

$$\text{length of } A_i \times (b_2 - a_2). \tag{6.65}$$

Then we can sum over $i$ and use the same result again to prove what we want.

(3) For any finite collection of disjoint rectangles contained in $[a_1, b_1) \times [a_2, b_2)$ we can use the same division process to show that we can add more disjoint rectangles to cover the whole big rectangle. Thus, from the preceeding result the sum of the areas must be less than or equal to $(b_1 - a_1)(b_2 - a_2)$. For a countable collection of disjoint rectangles the sum of the areas is therefore bounded above by this constant.

(4) Let the rectangles be $D_i$, $i = 1, \dots, N$ the union of which contains the rectangle $D$. Subdivide $D_1$ using all the endpoints of the intervals of $D$. Each of the resulting rectangles is either contained in $D$ or is disjoint from it. Replace $D_1$ by the (one in fact) subrectangle contained in $D$. Proceeding by induction we can suppose that the first $N-k$ of the rectangles are disjoint and all contained in $D$ and together all the rectangles cover $D$. Now look at the next one, $D_{N-k+1}$. Subdivide it using all the endpoints of the intervals for the earlier rectangles $D_1, \dots, D_k$ and $D$. After subdivision of $D_{N-k+1}$ each resulting rectangle is either contained in one of the $D_j$, $j \le N - k$ or is *not* contained in $D$. All these can be discarded and the result is to decrease $k$ by 1 (maybe increasing $N$ but that is okay). So, by induction we can decompose and throw away rectangles until what is left are disjoint and individually contained in $D$ but still cover. The sum of the areas of the remaining rectangles is precisely the area of $D$ by the previous result, so the sum of the areas must originally have been at least this large.

(5) Now, for a countable collection of rectangles covering $D = [a_1, b_1) \times [a_2, b_2)$ we proceed as in the one-dimensional case. First, we can assume that there is a fixed upper bound $C$ on the lengths of the sides. Make the $k$th rectangle a little larger by extending both the upper limits by $2^{-k}\delta$ where $\delta > 0$. The area increases, but by no more than $2C2^{-k}$. After extension the interiors of the countable collection cover the compact set $[a_1, b_1 - \delta] \times [a_2, b_1 - \delta]$. By compactness, a finite number of these open rectangles cover, and hence there semi-closed version, with the same endpoints, covers $[a_1, b_1 - \delta) \times$

$[a_2, b_1 - \delta)$. Applying the preceeding finite result we see that

$$\text{Sum of areas} + 2C\delta \geq \text{Area } D - 2C\delta. \tag{6.66}$$

Since this is true for all $\delta > 0$ the result follows.

□

I encourage you to go through the discussion of integrals of step functions – now based on rectangles instead of intervals – and see that everything we have done can be extended to the case of two dimensions. In fact if you want you can go ahead and see that everything works in $\mathbb{R}^n$!

*Problem 2.4*

(1) Show that any continuous function on $[0, 1]$ is the *uniform limit* on $[0, 1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into $2^n$ equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0, 1)$ can be written as the sum

$$\sum_i f_j(x) \ \forall \ x \in [0, 1) \tag{6.67}$$

where the $f_j$ are step functions and $\sum_j |f_j(x)| < \infty$ for all $x \in [0, 1)$.

(3) Conclude that any continuous function on $[0, 1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$.

*Solution.* (1) Since the real and imaginary parts of a continuous function are continuous, it suffices to consider a real continous function $f$ and then add afterwards. By the *uniform* continuity of a continuous function on a compact set, in this case $[0, 1]$, given $n$ there exists $N$ such that $|x - y| \leq 2^{-N} \Longrightarrow |f(x) - f(y)| \leq 2^{-n}$. So, if we divide into $2^N$ equal intervals, where $N$ depends on $n$ and we insist that it be non-decreasing as a function of $n$ and take the step function $f_n$ on each interval which is equal to $\min f = \inf f$ on the closure of the interval then

$$|f(x) - F_n(x)| \leq 2^{-n} \ \forall \ x \in [0, 1) \tag{6.68}$$

since this even works at the endpoints. Thus $F_n \to f$ uniformly on $[0, 1)$.

(2) Now just define $f_1 = F_1$ and $f_k = F_k - F_{k-1}$ for all $k > 1$. It follows that these are step functions and that

$$\sum_{k=1}^{n} = f_n. \tag{6.69}$$

Moreover, each interval for $F_{n+1}$ is a subinterval for $F_n$. Since $f$ can varying by no more than $2^{-n}$ on each of the intervals for $F_n$ it follows that

$$|f_n(x)| = |F_{n+1}(x) - F_n(x)| \leq 2^{-n} \ \forall \ n > 1. \tag{6.70}$$

Thus $\int |f_n| \leq 2^{-n}$ and so the series is absolutely summable. Moreover, it actually converges everywhere on $[0, 1)$ and uniformly to $f$ by (6.68).

(3) Hence $f$ is Lebesgue integrable.

(4) For some reason I did not ask you to check that

$$\int f = \int_0^1 f(x)dx \tag{6.71}$$

where on the right is the Riemann integral. However this follows from the fact that

$$\int f = \lim_{n\to\infty} \int F_n \tag{6.72}$$

and the integral of the step function is between the Riemann upper and lower sums for the corresponding partition of $[0, 1]$.

□