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18.102 Introduction to Functional Analysis
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Lecture 9. THURSDAY, MARCH 5

My attempts to distract you all during the test did not seem to work very well. Here is what I wrote up on the board, more or less. We will use Baire's theorem later (it is also known as 'Baire category theory' although it has nothing to do with categories in the modern sense).

This is a theorem about complete metric spaces – it could be included in 18.100B but the main applications are in Functional Analysis.

Theorem 4 (Baire). *If M is a non-empty complete metric space and $C_n \subset M$, $n \in \mathbb{N}$, are closed subsets such that*

$$(9.1) \quad M = \bigcup_n C_n$$

then at least one of the C_n 's has an interior point.

Proof. We can assume that the first set $C_1 \neq \emptyset$ since they cannot all be empty and dropping some empty sets does no harm. Let's assume the contrary of the desired conclusion, namely that each of the C_n 's has empty interior, hoping to arrive at a contradiction to (9.1) using the other properties. This means that an open ball $B(p, \epsilon)$ around a point of M (so it isn't empty) cannot be contained in any one of the C_n .

So, choose $p \in C_1$. Now, there must be a point $p_1 \in B(p, 1/3)$ which is not in C_1 . Since C_1 is closed there exists $\epsilon_1 > 0$, and we can take $\epsilon_1 < 1/3$, such that $B(p_1, \epsilon_1) \cap C_1 = \emptyset$. Continue in this way, choose $p_2 \in B(p_1, \epsilon_1/3)$ which is not in C_2 and $\epsilon_2 > 0$, $\epsilon_2 < \epsilon_1/3$ such that $B(p_2, \epsilon_2) \cap C_2 = \emptyset$. Here we use both the fact that C_2 has empty interior and the fact that it is closed. So, inductively there is a sequence p_i , $i = 1, \dots, k$ and positive numbers $0 < \epsilon_k < \epsilon_{k-1}/3 < \epsilon_{k-2}/3^2 < \dots < \epsilon_1/3^{k-1} < 3^{-k}$ such that $p_j \in B(p_{j-1}, \epsilon_{j-1}/3)$ and $B(p_j, \epsilon_j) \cap C_j = \emptyset$. Then we can add another p_{k+1} by using the properties of C_k – it has non-empty interior so there is some point in $B(p_k, \epsilon_k/3)$ which is not in C_{k+1} and then $B(p_{k+1}, \epsilon_{k+1}) \cap C_{k+1} = \emptyset$ where $\epsilon_{k+1} > 0$ but $\epsilon_{k+1} < \epsilon_k/3$. Thus, we have a sequence $\{p_k\}$ in M . Since $d(p_{k+1}, p_k) < \epsilon_k/3$ this is a Cauchy sequence, in fact

$$(9.2) \quad d(p_k, p_{k+l}) < \epsilon_k/3 + \dots + \epsilon_{k+l-1}/3 < 3^{-k}.$$

Since M is complete the sequence converges to a limit, $q \in M$. Notice however that $p_l \in B(p_k, 2\epsilon_k/3)$ for all $k > l$ so $d(p_k, q) \leq 2\epsilon_k/3$ which implies that $q \notin C_k$ for any k . This is the desired contradiction to (9.1).

Thus, at least one of the C_n must have non-empty interior. \square

One application of this, which we will get to later, is the *uniform boundedness principle* (which is just a theorem).

Theorem 5. *Let B be a Banach space and suppose that T_k is a sequence of bounded (i.e. continuous) linear operators $T_n : B \rightarrow V$ where V is a normed space. Suppose that for each $b \in B$ the set $\{T_n(b)\} \subset V$ is bounded (in norm of course) then $\sup_n \|T_n\| < \infty$.*

Proof. You can look it up, but it follows from an application of Baire's theorem to the sets

$$(9.3) \quad S_p = \{b \in B, \|b\| < 1, \|T_n b\|_V \leq p \forall n\}, p \in \mathbb{N}.$$

You can check that these are closed and that their union must be the closed ball of radius one around the origin in B (because of the assumption of ‘pointwise boundedness’) So, by Baire’s theorem one of them has non-empty interior. This means that for some p , some $v \in S_p$ and some $\delta > 0$,

$$(9.4) \quad w \in B, \|w\|_B \leq \delta \implies \|T_n(v+w)\|_V \leq p \quad \forall n.$$

Using the triangle inequality, and the fact that $\|T_n(v)\|_V \leq p$ this means

$$(9.5) \quad w \in B, \|w\|_B \leq \delta \implies \|T_n(w)\|_V \leq 2p \implies \|T_n\| \leq 2p/\delta$$

since the norm of the operator is $\sup\{\|T_n w\|_V; \|w\|_B = 1\}$

□

Why this should be useful we shall see!