18.102 Introduction to Functional Analysis
Spring 2009

Lecture 10. Tuesday, Mar 10

All of this is easy to find in the various reference notes and/or books so I will keep these notes very brief.

(1) Bessel's inequality

If in a preHilbert space H, e_i, $i = 1, \dots, N$ are orthonormal – so $(e_i, e_j) = \delta_{ij}$ then for any element $u \in H$, set

$$v = \sum_{i=1}^{N} (u, e_i) e_i \text{ then}$$
$$\|v\|_H^2 = \sum_{i=1}^{N} |(u, e_i)|^2 \le \|u\|_H^2, \tag{10.1}$$
$$(u - v) \perp e_i, \ i = 1, \dots, N.$$

The last statement follows immediately by computing $(u, e_j) = (v, e_j)$ and similarly $\|v\|^2$ can be computed directly. Then the inequality, which is Bessel's inequality, follows from Cauchy's inequality since from the last statement

$$\|v\|^2 = (v, v) = (v, u) + (v, v - u) = (v, u) = |(v, u)| \le \|v\|\|u\| \tag{10.2}$$

shows $\|v\| \le \|u\|$.

(2) Orthonormal bases:

Since in the inequality in (10.1) the right side is independent of N it follows that if $\{e_i\}_{i=1}^{\infty}$ is a countable orthonormal set then

$$\sum_{i=1}^{\infty} |(u, e_i)|^2 \le \|u\|_H^2. \tag{10.3}$$

From this it follows that the sequence

$$v_n = \sum_{i=1}^{n} (u, e_i) e_i \tag{10.4}$$

is Cauchy since if $m > n$,

$$\|v_n - v_m\|^2 = \sum_{n < j \le m} \|(u, e_j)|^2 \le \sum_{j=n+1}^{\infty} \|(u, e_j)|^2 \tag{10.5}$$

and the right side is small if n is large, independent of m.

Lemma 5. *If H is a Hilbert space – so now we assume completeness – and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence then for each $u \in H$,*

$$v = \sum_{j=1}^{\infty} (u, e_j) e_j \in H \tag{10.6}$$

converges and $(u - v) \perp e_j$ for all j.

Proof. The limit exists since the sequence is Cauchy and the space is complete. The orthogonality follows from the fact that $(u - v_n, e_j) = 0$ as soon as $n \ge j$ and

$$(u - v, e_j) = \lim_{n \to \infty} (u - v_n, e_j) = 0 \tag{10.7}$$

by continuity of the inner product (which follows from Cauchy's inequality). □

Now, we say an orthonormal sequence is complete, or is *and orthonormal basis* of H if $u \perp e_j = 0$ for all j implies $u = 0$. Then we see:-

Proposition 15. *If $\{e_j\|_{j=1}^\infty$ is an orthonormal basis in a Hilbert space H then*

$$u = \sum_{j=1}^{\infty} (u, e_j) e_j \ \forall \ u \in H. \tag{10.8}$$

Proof. From the lemma the series converges to v and $(u - v) \perp e_j$ for all j so by the assumed completeness, $u = v$ which is (10.8). □

(3) Gram-Schmidt

Theorem 6. *Every separable Hilbert space has an orthonormal basis.*

Proof. Take a countable dense subset – which can be arranged as a sequence $\{v_j\}$ and the existence of which is the definition of separability – and orthonormalize it. Thus if $v_1 \neq 0$ set $e_i = v_1/\|v_1\|$. Proceeding by induction we can suppose to have found for a given integer n elements e_i, $i = 1, \dots, m$, where $m \le n$, which are orthonormal and such that the linear span

$$\mathrm{sp}(e_1, \dots, e_m) = \mathrm{sp}(v_1, \dots, v_n). \tag{10.9}$$

To show the inductive step observe that if v_{n+1} is in the span(s) in (10.9) then the same e_i work for $n + 1$. So it follows that

$$w = v_{n+1} - \sum_{j=1}^{n} (v_{n+1}, e_j) e_j \neq 0 \text{ so } e_{m+1} = \frac{w}{\|w\|} \tag{10.10}$$

makes sense. Adding e_{m+1} gives the equality of the spans for $n + 1$.

Thus we may continue indefinitely. There are only two possibilities, either we get a finite set of e_i's or an infinite sequence. In either case this must be an orthonormal basis. That is we claim

$$H \ni u \perp e_j \ \forall \ j \Longrightarrow u = 0. \tag{10.11}$$

This uses the density of the v_n's. That is, there must exist a sequence w_j where each w_j is a v_n, such that $w_j \to u$ in H. Now, each each v_n, and hence each w_j, is a finite linear combination of e_k's so, by Bessel's inequality

$$\|w_j\|^2 = \sum_k |(w_j, e_k)|^2 = \sum_k |(u - w_j, e_k)|^2 \le \|u - w_j\|^2 \tag{10.12}$$

where $(u, e_j) = 0$ for all j has been used. Thus $\|w_j\| \to 0$ and $u = 0$. □

(4) Isomorphism to l^2

A finite dimensional Hilbert space is isomorphic to $\mathbb{C}^n$ with its standard inner product. Similarly from the result above

Proposition 16. *Any infinite-dimensional separable Hilbert space (over the complex numbers) is isomorphic to l^2, that is there exists a linear map*

$$T : H \longrightarrow L^2 \tag{10.13}$$

which is 1-1, onto and satisfies $(Tu, Tv)_{l^2} = (u, v)_H$ *and* $\|Tu\|_{l^2} = \|u\|_H$ *for all* u, $v \in H$.

Proof. Choose an orthonormal basis – which exists by the discussion above and set

$$Tu = \{(u, e_j)\|_{j=1}^{\infty}. \tag{10.14}$$

This maps H into l^2 by Bessel's inequality. Moreover, it is linear since the entries in the sequence are linear in u. It is 1-1 since $Tu = 0$ implies $(u, e_j) = 0$ for all j implies $u = 0$ by the assumed completeness of the orthonormal basis. It is surjective since if $\{c_j\}_{j=1}^{\infty}$ then

$$u = \sum_{j=1}^{\infty} c_j e_j \tag{10.15}$$

converges in H. This is the same argument as above – the sequence of partial sums is Cauchy by Bessel's inequality. Again by continuity of the inner product, $Tu = \{c_j\}$ so T is surjective.

The equality of the norms follows from equality of the inner products and the latter follows by computation for finite linear combinations of the e_j and then in general by continuity. □

Problem set 5, Due 11AM Tuesday 17 Mar.

You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases} \tag{10.16}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Problem 5.2 Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases} \tag{10.17}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{10.18}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{10.19}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N],\ x \in [-L, L] \\ N & \text{if } h_n(x) > N,\ x \in [-L, L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L, L] \end{cases}. \tag{10.20}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Problem 5.3 Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space.

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g_L^{(N)}$ and $h_n^{(N)}$ by (10.17), (10.18) and (10.20).

(2) Show using the sequence $h_{n,L}^{(N)}$ for fixed N and L that $g_L^{(N)}$ and $(g_L^{(N)})^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int |(h_{n,L}^{(N)})^2 - (g_L^{(N)})^2| \to 0$ as $n \to \infty$.

(3) Show that $(g_L)^2 \in \mathcal{L}^1(\mathbb{R})$ and that $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.

(4) Show that $\int |(g_L)^2 - f^2| \to 0$ as $L \to \infty$.

(5) Show that f, $g \in \mathcal{L}^2(\mathbb{R})$ then $fg \in \mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg| \le \int |fg| \le \|f\|_{L^2} \|g\|_{L^2},\ \|f\|_{L^2}^2 = \int |f|^2. \tag{10.21}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Problem 5.4
Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{10.22}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{10.23}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.
(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{10.24}$$

Problem 5.5 In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{10.25}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{10.26}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{10.27}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H\ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{10.28}$$

Solutions to Problem set 4

Just to compensate for last week, I will make this problem set too short and easy!

Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2)\ \forall\ u,v\in H. \tag{10.29}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u,v)=\frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2+i\|u+iv\|^2-i\|u-iv\|^2\right)! \tag{10.30}$$

Solution: Setting $u=v$, even without the parallelogram law,

$$(u,u)=\frac{1}{4}\left\|2u\|^2+i\|(1+i)u\|^2-i\|(1-i)u\|^2\right)=\|u\|^2. \tag{10.31}$$

So the point is that the parallelogram law shows that (u,v) is indeed an Hermitian inner product. Taking complex conjugates and using properties of the norm, $\|u+iv\|=\|v-iu\|$ etc

$$\overline{(u,v)}=\frac{1}{4}\left(\|v+u\|^2-\|v-u\|^2-i\|v-iu\|^2+i\|v+iu\|^2\right)=(v,u). \tag{10.32}$$

Thus we only need check the linearity in the first variable. This *is* a little tricky! First compute away. Directly from the identity $(u,-v)=-(u,v)$ so $(-u,v)=-(u,v)$ using (10.32). Now,

$$\begin{aligned}
(2u,v)=&\frac{1}{4}\big(\|u+(u+v)\|^2-\|u+(u-v)\|^2\\
&+i\|u+(u+iv)\|^2-i\|u+(u-iv)\|^2\big)\\
=&\frac{1}{2}\big(\|u+v\|^2+\|u\|^2-\|u-v\|^2-\|u\|^2\\
&+i\|(u+iv)\|^2+i\|u\|^2-i\|u-iv\|^2-i\|u\|^2\big)\\
&-\frac{1}{4}\left(\|u-(u+v)\|^2-\|u-(u-v)\|^2+i\|u-(u+iv)\|^2-i\|u-(u-iv)\|^2\right)\\
=&2(u,v).
\end{aligned} \tag{10.33}$$

Using this and (10.32), for any u, u' and v,

$$\begin{aligned}
(u+u',v)=&\frac{1}{2}(u+u',2v)\\
=&\frac{1}{2}\frac{1}{4}\big(\|(u+v)+(u'+v)\|^2-\|(u-v)+(u'-v)\|^2\\
&+i\|(u+iv)+(u-iv)\|^2-i\|(u-iv)+(u'-iv)\|^2\big)\\
=&\frac{1}{4}\big(\|u+v\|+\|u'+v\|^2-\|u-v\|-\|u'-v\|^2\\
&+i\|(u+iv)\|^2+i\|u-iv\|^2-i\|u-iv\|-i\|u'-iv\|^2\big)\\
&-\frac{1}{2}\frac{1}{4}\big(\|(u+v)-(u'+v)\|^2-\|(u-v)-(u'-v)\|^2\\
&+i\|(u+iv)-(u-iv)\|^2-i\|(u-iv)=(u'-iv)\|^2\big)\\
=&(u,v)+(u',v).
\end{aligned} \tag{10.34}$$

Using the second identity to iterate the first it follows that $(ku, v) = k(u, v)$ for any u and v and any positive integer k. Then setting $nu' = u$ for any other positive integer and $r = k/n$, it follows that

$$(ru, v) = (ku', v) = k(u', v) = rn(u', v) = r(u, v) \tag{10.35}$$

where the identity is reversed. Thus it follows that $(ru, v) = r(u, v)$ for any rational r. Now, from the definition both sides are continuous in the first element, with respect to the norm, so we can pass to the limit as $r \to x$ in $\mathbb{R}$. Also directly from the definition,

$$(iu, v) = \frac{1}{4}\left(\|iu + v\|^2 - \|iu - v\|^2 + i\|iu + iv\|^2 - i\|iu - iv\|^2\right) = i(u, v) \tag{10.36}$$

so now full linearity in the first variable follows and that is all we need.

Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{10.37}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (10.37) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (10.37) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{10.38}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i},\ \|u\|_H = \|Tu\|_{\mathbb{C}^n}\ \forall\ u, v \in H. \tag{10.39}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Solution: Since H is assumed to be finite dimensional, it has a basis v_i, $i = 1, \dots, n$. This basis can be replaced by an orthonormal basis in n steps. First replace v_1 by $e_1 = v_1/\|v_1\|$ where $\|v_1\| \neq 0$ by the linear indepedence of the basis. Then replace v_2 by

$$e_2 = w_2/\|w_2\|,\ w_2 = v_2 - \langle v_2, e_1\rangle e_1. \tag{10.40}$$

Here $w_2 \perp e_1$ as follows by taking inner products; w_2 cannot vanish since v_2 and e_1 must be linearly independent. Proceeding by finite induction we may assume that we have replaced $v_1, v_2, \dots, v_k$, $k < n$, by $e_1, e_2, \dots, e_k$ which are orthonormal and span the same subspace as the v_i's $i = 1, \dots, k$. Then replace v_{k+1} by

$$e_{k+1} = w_{k+1}/\|w_{k+1}\|,\ w_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}, e_i\rangle e_i. \tag{10.41}$$

By taking inner products, $w_{k+1} \perp e_i$, $i = 1, \dots, k$ and $w_{k+1} \neq 0$ by the linear independence of the v_i's. Thus the orthonormal set has been increased by one element preserving the same properties and hence the basis can be orthonormalized.

Now, for each $u \in H$ set

$$c_i = \langle u, e_i\rangle. \tag{10.42}$$

It follows that $U = u - \sum\limits_{i=1}^{n} c_i e_i$ is orthogonal to all the e_i since

$$\langle u, e_j \rangle = \langle u, e_j \rangle - \sum_i c_i \langle e_i, e_j \rangle = \langle u.e_j \rangle - c_j = 0. \tag{10.43}$$

This implies that $U = 0$ since writing $U = \sum\limits_i d_i e_i$ it follows that $d_i = \langle U, e_i \rangle = 0$.

Now, consider the map (10.38). We have just shown that this map is injective, since $Tu = 0$ implies $c_i = 0$ for all i and hence $u = 0$. It is linear since the c_i depend linearly on u by the linearity of the inner product in the first variable. Moreover it is surjective, since for any $c_i \in \mathbb{C}$, $u = \sum\limits_i c_i e_i$ reproduces the c_i through (10.42). Thus T is a linear isomorphism and the first identity in (10.39) follows by direct computation:-

$$\begin{aligned} \sum_{i=1}^{n} (Tu)_i \overline{(Tv)_i} &= \sum_i \langle u, e_i \rangle \\ &= \langle u, \sum_i \langle v, e_i \rangle e_i \rangle \\ &= \langle u, v \rangle. \end{aligned} \tag{10.44}$$

Setting $u = v$ shows that $\|Tu\|_{\mathbb{C}^n} = \|u\|_H$.

Now, we know that $\mathbb{C}^n$ is complete with its standard norm. Since T is an isomorphism, it carries Cauchy sequences in H to Cauchy sequences in $\mathbb{C}^n$ and T^{-1} carries convergent sequences in $\mathbb{C}^n$ to convergent sequences in H, so every Cauchy sequence in H is convergent. Thus H is complete.