

MIT OpenCourseWare
<http://ocw.mit.edu>

18.102 Introduction to Functional Analysis
Spring 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

Lecture 11. THURSDAY, 12 MAR

Quite a lot of new material, but all of it in the various notes and books. So, I will keep it brief.

(1) Convex sets and length minimizer

The following result does not need the hypothesis of separability of the Hilbert space and allows us to prove the subsequent results – especially Riesz' theorem – in full generality.

Proposition 17. *If $C \subset H$ is a subset of a Hilbert space which is*

(a) *Non-empty*

(b) *Closed*

(c) *Convex, in the sense that $v_1, v_2 \in C$ implies $\frac{1}{2}(v_1 + v_2) \in C$*

then there exists a unique element $v \in C$ closest to the origin, i.e. such that

$$(11.1) \quad \|v\|_H = \inf_{u \in C} \|u\|_H.$$

Proof. By definition of inf there must exist a sequence $\{v_n\}$ in C such that $\|v_n\| \rightarrow d = \inf_{u \in C} \|u\|_H$. We show that v_n converges and that the limit is the point we want. The parallelogram law can be written

$$(11.2) \quad \|v_n - v_m\|^2 = 2\|v_n\|^2 + 2\|v_m\|^2 - 4\|(v_n + v_m)/2\|^2.$$

Since $\|v_n\| \rightarrow d$, given $\epsilon > 0$ if N is large enough then $n > N$ implies $2\|v_n\|^2 < 2d^2 + \epsilon^2/2$. By convexity, $(v_n + v_m)/2 \in C$ so $\|(v_n + v_m)/2\|^2 \geq d^2$. Combining these estimates gives

$$(11.3) \quad n, m > N \implies_{u \in C} \|u\|_H \leq 4d^2 + \epsilon^2 - 4d^2$$

so $\{v_n\}$ is Cauchy. Since H is complete, $v_n \rightarrow v \in C$ since C is closed. Moreover, the distance is continuous so $\|v\|_H = \lim_{n \rightarrow \infty} \|v_n\| = d$.

Thus v exists and uniqueness follows again from the parallelogram law. If v and v' are two points in C with $\|v\| = \|v'\| = d$ then $(v + v')/2 \in C$ so

$$(11.4) \quad \|v - v'\|^2 = 2\|v\|^2 + 2\|v'\|^2 - 4\|(v + v')/2\|^2 \leq 0 \implies v = v'.$$

□

(2) Orthocomplements

Proposition 18. *If $W \subset H$ is a linear subspace of a Hilbert space the*

$$(11.5) \quad W^\perp = \{u \in H; (u, w) = 0 \forall w \in W\}$$

is also a linear subspace and $W \cap W^\perp = \{0\}$. If W is also closed then

$$(11.6) \quad H = W \oplus W^\perp$$

meaning that any $u \in H$ has a unique decomposition $u = w + w^\perp$ where $w \in W$ and $w^\perp \in W^\perp$.

Proof. That W^\perp defined by (11.5) is a linear subspace follows from the linearity of the condition defining it. If $u \in W^\perp$ and $u \in W$ $u \perp u$ by the definition so $(u, u) = \|u\|^2 = 0$ and $u = 0$.

Now, suppose W is closed. If $W = H$ then $W^\perp = \{0\}$ and there is nothing to show. So consider $u \in H$, $u \notin W$. Consider

$$(11.7) \quad C = u + W = \{u' \in H; u' = u + w, w \in W\}.$$

Then C is closed, since a sequence in it is of the form $u'_n = u + w_n$ where w_n is a sequence in W and u'_n converges if and only if w_n converges. Now, C is non-empty, since $u \in C$ and it is convex since $u' = u + w'$ and $u'' = u + w''$ in C implies $(u' + u'')/2 = u + (w' + w'')/2 \in C$.

Thus the length minimization result above applies and there exists a unique $v \in C$ such that $\|v\| = \inf_{u' \in C} \|u'\|$. The claim is that this v is perpendicular to W – draw a picture in two real dimensions! To see this consider an arbitrary point $w \in W$ and $\lambda \in \mathbb{C}$ then $v + \lambda w \in C$ and

$$(11.8) \quad \|v + \lambda w\|^2 = \|v\|^2 + 2\operatorname{Re}(\lambda(v, w)) + |\lambda|^2\|w\|^2.$$

Choose $\lambda = te^{i\theta}$ where the phase is chosen so that $e^{i\theta}(v, w) = |(v, w)| \geq 0$. Then the fact that $\|v\|$ is minimal means that

$$(11.9) \quad t(2|(v, w)| + t\|w\|^2) \geq 0 \quad \forall t \in \mathbb{R} \implies |(v, w)| = 0$$

which is what we wanted to show.

Thus indeed, give $u \notin W$ we have constructed $v \in W^\perp$ such that $u = v + w$, $w \in W$. This is (11.6) with the uniqueness of the decomposition already shown since it reduces to 0 having only the decomposition $0 + 0$ and this in turn is $W \cap W^\perp = \{0\}$. \square

(3) Riesz' theorem

The most important application of these results is to prove Riesz' representation theorem (for Hilbert space, there is another one to do with measures).

Theorem 7. *If H is a Hilbert space then any continuous linear functional $T : H \rightarrow \mathbb{C}$ there exists a unique element $\phi \in H$ such that*

$$(11.10) \quad T(u) = (u, \phi) \quad \forall u \in H.$$

Proof. (a) Here is the proof I gave quickly in Lecture 10, not using the preceding Lemma. If T is the zero functional then $w = 0$ satisfies (11.10). Otherwise there exists some $u' \in H$ such that $T(u') \neq 0$ and then $u \in H$, namely $u = u'/T(u')$, such that $T(u) = 1$. Thus

$$(11.11) \quad C = \{u \in H; T(u) = 1\} = T^{-1}(\{1\})$$

is non-empty. The continuity of T and the second form shows that C is closed, as the inverse image of a closed set under a continuous map. Moreover C is convex since

$$(11.12) \quad T((u + u')/2) = (T(u) + T(u'))/2.$$

Thus, there exists an element $v \in C$ of minimal length. As in the proof above, it follows that $\|v + \lambda w\|^2 \geq \|v\|^2$ for all $w \in W$ and $\lambda \in \mathbb{C}$ this implies that $v \in W^\perp$. Now continue as in the proof below.

(b) Here is the proof I gave in Lecture 11 using the orthocomplement above. Since T is continuous the null space

$$(11.13) \quad W = T^{-1}(\{0\}) = \{u \in H; T(u) = 0\}$$

is a closed linear subspace. Thus

$$(11.14) \quad H = W \oplus W^\perp$$

by Proposition 18 above. Now, if $T = 0$ is the zero functional then $W = H$ and $W^\perp = \{0\}$ and $w = 0$ works in (11.10). Otherwise, $W^\perp \ni v'$, which is not in W , i.e. has $T(v') \neq 0$ and hence $v \in W^\perp$ with $T(v) = 1$. Then for any $u \in H$,

$$(11.15) \quad u - T(u)v \text{ satisfies } T(u - T(u)v) = T(u) - T(u)T(v) = 0 \longrightarrow u = w + T(u)v, \quad w \in W.$$

Then, $(u, v) = T(u)\|v\|^2$ since $(w, v) = 0$. Thus if $\phi = v/\|v\|^2$ then

$$(11.16) \quad u = w + (u, \phi)v \implies T(u) = (u, \phi)T(v) = (u, \phi).$$

□

(4) Adjoint of bounded operators. As an application of Riesz' theorem I showed that any bounded linear operator on a Hilbert space

$$(11.17) \quad A : H \longrightarrow H, \quad \|Hu\|_H \leq C\|u\|_H \quad \forall u \in H$$

has a unique adjoint operator. That is there exists a unique bounded linear operator $A^* : H \longrightarrow H$ such that

$$(11.18) \quad (Au, v)_H = (u, A^*v) \quad \forall u, v \in H.$$

To see the existence of A^*v we need to work out what A^*v should be for each fixed $v \in H$. So, fix v in the desired identity (11.18), which is to say consider

$$(11.19) \quad H \ni u \longrightarrow (Au, v) \in \mathbb{C}.$$

This is a linear map and it is clearly bounded, since

$$(11.20) \quad |(Au, v)| \leq \|Au\|_H \|v\|_H \leq (C\|v\|_H)\|u\|_H.$$

Thus it is a continuous linear functional on H which depends on v . In fact it is just the composite of two continuous linear maps

$$(11.21) \quad H \xrightarrow{u \mapsto Au} H \xrightarrow{w \mapsto (w, v)} \mathbb{C}.$$

By Riesz theorem there exists a unique element in H , which we can denote A^*v (since it only depends on v) such that

$$(11.22) \quad (Au, v) = (u, A^*v) \quad \forall u \in H.$$

Now this defines the map $A^*H \longrightarrow H$ but we need to check that it is linear and continuous. Linearity follows from the uniqueness part of Riesz' theorem. Thus if $v_1, v_2 \in H$ and $c_1, c_2 \in \mathbb{C}$ then

$$(11.23) \quad \begin{aligned} (Au, c_1v_1 + c_2v_2) &= \overline{c_1}(Au, v_1) + \overline{c_2}(Au, v_2) \\ &= \overline{c_1}(u, A^*v_1) + \overline{c_2}(u, A^*v_2) = (u, c_1A^*v_1 + c_2A^*v_2) \end{aligned}$$

where we have used the definitions of A^*v_1 and A^*v_2 - by uniqueness we must have $A^*(c_1v_1 + c_2v_2) = c_1A^*v_1 + c_2A^*v_2$.

Since we know the optimality of Cauchy's inequality

$$(11.24) \quad \|v\|_H = \sup_{\|u\|=1} |(u, v)|$$

(do we? If not set $u = v/\|v\|$ to see it.) it follows that

$$(11.25) \quad \|A^*v\| = \sup_{\|u\|=1} |(u, A^*v)| = \sup_{\|u\|=1} |(Au, v)| \leq \|A\|\|v\|.$$

So in fact

$$(11.26) \quad \|A^*\| \leq \|A\|.$$

In fact it is immediately the case that $(A^*)^* = A$ so the reverse equality also holds and so

$$(11.27) \quad \|A^*\| = \|A\|.$$