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18.102 Introduction to Functional Analysis
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Lecture 12. Tuesay, Mar 17: Compactness and weak convergence

A subset in a general metric space is one with the property that any sequence in it has a convergent subsequence, with its limit in the set. You will recall with pleasure no doubt the equivalence of this condition to the (more general since it makes good sense in an arbitrary topological space) equivalence of this with the covering condition, that *any* open cover of the set has a finite subcover. So, in a separable Hilbert space the notion of a compact set is already fixed. We want to characterize it – in the problems this week you will be asked to prove several characterizations.

A general result in a metric space is that any compact set is both closed and bounded, so this must be true in a Hilbert space. The Heine-Borel theorem gives a converse to this, $\mathbb{R}^n$ or $\mathbb{C}^n$ (and hence in any finite dimensional normed space) any closed and bounded set is compact. Also recall that the convergence of a sequence in $\mathbb{C}^n$ is equivalent to the convergence of the n sequences given by its components and this is what is used to pass first from $\mathbb{R}$ to $\mathbb{C}$ and then to $\mathbb{C}^n$. All of this fails in infinite dimensions and we need some condition in addition to being bounded and closed for a set to be compact.

To see where this might come from, observe that a set, S, consisting of the points of a convergent sequence, $s : \mathbb{N} \longrightarrow M$, together with its limit, s, in any metric space is always compact. The set here is the image of the sequence, thought of as a map from the integers into the metric space, together with the limit (which might of course already be in the image). Certainly this set is bounded, since the distance from the intial point is certainly bounded. Moreover it is closed, although you might need to think about this for a minute. A sequence in the set which is the image of another sequence consists of elements of the original sequence in any order and maybe repeated at will. Since the original sequence may itself have reapeated points, the labelling of points is by no means unique. However S is closed since $M \setminus S$ is open – a point in $p \in M \setminus S$ is at a finite no-zero distance, $d(p, s)$ from the limit so $B(p, d(p, s)/2)$ can contain only finitely many elements of S hence a smaller open ball does not meet it.

Lemma 6. *The image of a convergent sequence in a Hilbert space is a set with equi-small tails with respect to any orthonormal sequence, i.e. if* e_k *is an othonormal sequence and* $u_n \to u$ *is a convergent sequence then given* $\epsilon > 0$ *there exists* N *such that*

$$\sum_{k>N} |(u_n, e_k)|^2 < \epsilon^2 \ \forall \ n. \tag{12.1}$$

Proof. Bessel's inequality shows that for any $u \in \mathcal{H}$,

$$\sum_{k} |(u, e_k)|^2 \le \|u\|^2. \tag{12.2}$$

The convergence of this series means that (12.1) can be arranged for any single element u_n or the limit u by choosing N large enough, thus given $\epsilon > 0$ we can choose N' so that

$$\sum_{k>N'} |(u, e_k)|^2 < \epsilon^2/2. \tag{12.3}$$

In fact, for any orthonormal sequence such as e_k – whether complete or not,

$$P : \mathcal{H} \ni u \longmapsto Pu = \sum_k (u, e_k) e_k \in \mathcal{H} \tag{12.4}$$

is continuous and in fact has norm at most one. Indeed from Bessel's inequality, $\|Pu\|^2 \le \|u\|^2$. Now, applying this to

$$P_N u = \sum_{k>N} (u, e_k) e_k \tag{12.5}$$

the convergence $u_n \to u$ implies the convergence in norm $\|P_N u_n\| \to \|P_N u\|$ and so

$$\sum_{k>N'} |(u, e_k)|^2 < \epsilon^2. \tag{12.6}$$

So, we have arranged (12.1) for $n > n'$ with $N = N'$. Of course, this estimate remains valid if N is increased, and we may arrange it for $n \le n'$ by chossing N large enough. Thus indeed (12.1) holds for all n if N is chosen large enough. □

This suggest one useful characterization of compact sets in a separable Hilbert space.

Proposition 19. *A set $K \subset \mathcal{H}$ in a separabel Hilbert space is compact if and only if it is bounded, closed and has equi-small tails with respect to any one orthonormal basis.*

Proof. We already know that a compact set is closed and bounded. Suppose the equi-smallness of tails condition fails with respect to some orthonormal basis e_k. This means that for some $\epsilon > 0$ and all N there is an element $u_N \in K$ such that

$$\sum_{k>N} |(u_N, e_k)|^2 \ge \epsilon^2. \tag{12.7}$$

Then the sequence $\{u_N\}$ can have no convergent subsequence, since this would contradict the Lemma we have just proved, hence K is not compact in this case.

Thus we have proved the equi-smallness of tails condition to be necessary for the compactness of a closed, bounded set. So, it remains to show that it is sufficient. So, suppose K is closed, bounded and satisfies the equi-small tails condition with respect to an orthonormal basis e_k and $\{u_n\}$ is a sequence in K. We only need show that $\{u_n\}$ has a Cauchy subsequence, since this will converge ($\mathcal{H}$ being complete) and the limit will be in K (since it is closed). Now, consider each of the sequences of coefficients (u_n, e_k) in $\mathbb{C}$. Here k is fixed. This sequence is bounded:

$$|(u_n, e_k)| \le \|u_n\| \le C \tag{12.8}$$

by the boundedness of K. So, by the Heine-Borel theorem, there is a subsequence of u_{n_l} such that (u_{n_l}, e_k) converges as $l \to \infty$.

We can apply this argument for each $k = 1, 2, \ldots$. First extracting a subsequence of $\{u_n\}$ so that the sequence (u_n, e_1) converges 'along this subsequence'. Then extract a subsequence of this subsequence so that (u_n, e_2) *also* converges along this sparser subsequence, and continue inductively. Then pass to the 'diagonal' subsequence of $\{u_n\}$ which has kth entry the kth term in the kth subsequence. It is 'eventually' a subsequence of each of the subsequences previously constructed – meaning it coincides with a subsequence for some point onward (namely the

kth term onward for the kth subsquence). Thus, for this subsequence *each* of the (u_{n_l}, e_k) converges.

Now, let's relabel this subsequence v_n for simplicity of notation and consider Bessel's identity (the orthonormal set e_k is complete by assumption) for the difference

$$\begin{aligned} \|v_n - v_{n+l}\|^2_{\mathcal{H}} &= \sum_{k \le N} |(v_n - v_{n+l}, e_k)|^2 + \sum_{k > N} |(v_n - v_{n+l}, e_k)|^2 \\ &\le \sum_{k \le N} |(v_n - v_{n+l}, e_k)|^2 + 2 \sum_{k > N} |(v_n, e_k)|^2 + 2 \sum_{k > N} |(v_{n+l}, e_k)|^2 \end{aligned} \tag{12.9}$$

where the parallelogram law on $\mathbb{C}$ has been used. To make this sum less than ϵ^2 we may choose N so large that the last two terms are less than $\epsilon^2/2$ and this may be done for all n and l by the equi-smallness of the tails. Now, choose n so large that each of the terms in the first sum is less than $\epsilon^2/2N$, for all $l > 0$ using the Cauchy condition on each of finite number of sequence (v_n, e_k). Thus, $\{v_n\}$ is a Cauchy subsequence of $\{u_n\}$ and hence as already noted convergent in K. Thus K is indeed compact. □

It is convenient to formalize the idea that each of the (u_n, e_k), the sequence of coefficients of the Fourier-Bessel series, should converge.

Definition 6. A sequence, $\{u_n\}$, in a Hilbert space, $\mathcal{H}$, is said to *converge weakly* to an element $u \in \mathcal{H}$ if it is bounded in norm and $(u_j, v) \to (u, v)$ converges in $\mathbb{C}$ for each $v \in \mathcal{H}$. This relationship is written

$$u_n \rightharpoonup u. \tag{12.10}$$

In fact as we shall see next time, the assumption that $\|u_n\|$ is bounded and that u exists are both unnecessary. That is, a sequence converges weakly if and only if (u_n, v) converges in $\mathbb{C}$ for each $v \in \mathcal{H}$. Conversely, there is no harm in assuming it is bounded and that the 'weak limit' $u \in \mathcal{H}$ exists. Note that the weak limit is unique since if u and u' both have this property then $(u - u', v) = \lim_{n\to\infty}(u_n, v) - \lim_{n\to\infty}(u_n, v) = 0$ for all $v \in \mathcal{H}$ and setting $v = u - u'$ it follows that $u = u'$.

Lemma 7. *A (strongly) convergent sequence is weakly convergent with the same limit.*

Proof. This is the continuity of the inner product. If $u_n \to u$ then

$$|(u_n, v) - (u, v)| \le \|u_n - u\|\|v\| \to 0 \tag{12.11}$$

for each $v \in \mathcal{H}$ shows weak convergence. □

Now, there is a couple of things I will prove here and leave some more to you for the homework.

Lemma 8. *For a bounded sequence in a separable Hilbert space, weak convergence is equivalent to component convergence with respect to an orthonormal basis.*

Proof. Let e_k be an orthonormal basis. Then if u_n is weakly convergent it follows immediately that $(u_n, e_k) \to (u, e_k)$ converges for each k. Conversely, suppose this is true for a bounded sequence, just that $(u_n, e_k) \to c_k$ in $\mathbb{C}$ for each k. The norm boundedness and Bessel's inequality show that

$$\sum_{k \le p} |c_k|^2 = \lim_{n\to\infty} \sum_{k \le p} |(u_n, e_k)|^2 \le C^2 \sup_n \|u_n\|^2 \tag{12.12}$$

for all p. Thus in fact $\{c_k\} \in l^2$ and hence

$$u = \sum_k w_k e_k \in \mathcal{H} \tag{12.13}$$

by the completeness of $\mathcal{H}$. Clearly $(u_n, e_k) \to (u, e_k)$ for each k. It remains to show thta $(u_n, v) \to (u, v)$ for all $v \in \mathcal{H}$. This is certainly true for any finite linear combination of the e_k and for a general v we can write

$$\begin{aligned}(u_n, v) - (u, v) = (u_n, v_p) - (u, v_p) + (u_n, v - v_p) - (u, v - v_p) \Longrightarrow \\ |(u_n, v) - (u, v)| = |(u_n, v_p) - (u, v_p)| + 2C\|v - v_p\|\end{aligned} \tag{12.14}$$

where $v_p = \sum\limits_{k \le p} (v, e_k) e_k$ is a finite part of the Fourier-Bessel series for v and C is a bound for $\|u_n\|$. Now the convergence $v_p \to v$ implies that the last term in (12.14) can be made small by choosing p large, independent of n. Then the second last term can be made small by choosing n large since v_p is a finite linear combination of the e_k. Thus indeed, $(u_n, v) \to (u, v)$ for all $v \in \mathcal{H}$ an it follows that u_n converges weakly to u. □

Proposition 20. *Any bounded sequence* $\{u_n\}$ *in a separable Hilbert space has a weakly convergent subsequence.*

This can be thought of as an analogue in infinite dimensions of the Heine-Borel theorem if you say 'a bounded closed subset of a separable Hilbert space is *weakly* compact'.

Proof. Choose an orthonormal basis e_k and apply the procedure in the proof of Proposition 19 to extract a subsequence of the given bounded sequence such that (u_{n_p}, e_k) converges for each k. Now apply the preceeding Lemma to conclude that this subsequence converges weakly. □

Lemma 9. *For a weakly convergent sequence* $u_n \rightharpoonup u$

$$\|u\| \le \liminf \|u_n\|. \tag{12.15}$$

Proof. Choose an orthonormal basis e_k and observe that

$$\sum_{k \le p} \|u, e_k\|^2 = \lim_{n \to \infty} \|u_n, e_k\|^2. \tag{12.16}$$

Now the sequence on the right is bounded by $\|u_n\|^2$ independently of p so

$$\sum_{k \le p} \|u, e_k\|^2 \le \liminf_n \|u_n\|^2 \tag{12.17}$$

by the definition of $\liminf$. Now, take $p \to \infty$ to conclude that

$$\|u\|^2 \le \liminf_n \|u_n\|^2 \tag{12.18}$$

from which (12.15) follows. □

Problems 6: Due 11AM Tuesday, 31 Mar

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpfult. An example being the old hint for Problem 6.2!

Problem 6.1 Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 6.2 Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{12.19}$$

Hint:- To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{12.20}$$

Problem 6.3 Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{12.21}$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v'_n\}$ in D_N where v'_n is the closest point in D_N to v_n. Show that v'_n is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 6.4 Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

Problem 6.5 Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$\langle Au_1, u_2\rangle_{H_2} = \langle u_1, A^*u_2\rangle_{H_1} \ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{12.22}$$

Solutions to Problem set 5

You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases} \tag{12.23}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Solution. If χ_L is the characteristic function of $[-N, N]$ then $f_L = f\chi_L$. If f_n is an absolutely summable series of step functions converging a.e. to f then $f_n\chi_L$ is absolutely summable, since $\int |f_n\chi_L| \le \int |f_n|$ and converges a.e. to f_L, so $f_L \int \mathcal{L}^1(\mathbb{R})$. Certainly $|f_L(x) - f(x)| \to 0$ for each x as $L \to \infty$ and $|f_L(x) - f(x)| \le |f_l(x)| + |f(x)| \le 2|f(x)|$ so by Lebesgue's dominated convergence, $\int |f - f_L| \to 0$.

Problem 5.2 Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases} \tag{12.24}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{12.25}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{12.26}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N],\ x \in [-L, L] \\ N & \text{if } h_n(x) > N,\ x \in [-L, L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L, L] \end{cases}. \tag{12.27}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Solution:

(1) By definition $g_L^{(N)} = \max(-N\chi_L, \min(N\chi_L, g_L))$ where χ_L is the characteristic funciton of $-[L, L]$, thus it is in $\mathcal{L}^1(\mathbb{R})$.

(2) Clearly $g_L^{(N)}(x) \to g_L(x)$ for every x and $|g_L^{(N)}(x)| \le |g_L(x)|$ so by Dominated Convergence, $g_L^{(N)} \to g_L$ in L^1, i.e. $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$ since the sequence converges to 0 pointwise and is bounded by $2|g(x)|$.

(3) Let $S_{L,n}$ be a sequence of step functions converging a.e. to g_L – for example the sequence of partial sums of an absolutely summable series of step functions converging to g_L which exists by the assumed integrability.

Then replacing $S_{L,n}$ by $S_{L,n}\chi_L$ we can assume that the elements all vanish outside $[-N,N]$ but still have convergence a.e. to g_L. Now take the sequence

$$h_n(x) = \begin{cases} S_{k,n-k} & \text{on } [k,-k]\setminus[(k-1),-(k-1)],\ 1\le k\le n, \\ 0 & \text{on } \mathbb{R}\setminus[-n,n]. \end{cases} \tag{12.28}$$

This is certainly a sequence of step functions – since it is a finite sum of step functions for each n – and on $[-L,L]\setminus[-(L-1),(L-1)]$ for large integral L is just $S_{L,n-L}\to g_L$. Thus $h_n(x)\to f(x)$ outside a countable union of sets of measure zero, so also almost everywhere.

(4) This is repetition of the first problem, $h^{(N)}_{n,L}(x)\to g^{(N)}_L$ almost everywhere and $|h^{(N)}_{n,L}|\le N\chi_L$ so $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and $\int|h^{(N)}_{n,L}-g^{(N)}_L|\to 0$ as $n\to\infty$.

Problem 5.3 Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space – since it is rather central to the course I wanted you to go through the details carefully!

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g^{(N)}_L$ and $h^{(N)}_n$ by (12.24), (12.25) and (12.27).
(2) Show using the sequence $h^{(N)}_{n,L}$ for fixed N and L that $g^{(N)}_L$ and $(g^{(N)}_L)^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int|(h^{(N)}_{n,L})^2-(g^{(N)}_L)^2|\to 0$ as $n\to\infty$.
(3) Show that $(g_L)^2\in\mathcal{L}^1(\mathbb{R})$ and that $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) Show that $\int|(g_L)^2-f^2|\to 0$ as $L\to\infty$.
(5) Show that $f,\ g\in\mathcal{L}^2(\mathbb{R})$ then $fg\in\mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg|\le\int|fg|\le\|f\|_{L^2}\|g\|_{L^2},\ \ \|f\|^2_{L^2}=\int|f|^2. \tag{12.29}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R})=\mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Solution:

(1) Done. I think it should have been $h^{(N)}_{n,L}$.
(2) We already checked that $g^{(N)}_L\in\mathcal{L}^1(\mathbb{R})$ and the same argument applies to $(g^{(N)}_L)$, namely $(h^{(N)}_{n,L})^2\to g^{(N)}_L$ almost everywhere and both are bounded by $N^2\chi_L$ so by dominated convergence

$$\begin{gathered}(h^{(N)}_{n,L})^2\to g^{(N)}_L)^2\le N^2\chi_L \text{ a.e. } \Longrightarrow g^{(N)}_L)^2\in\mathcal{L}^1(\mathbb{R}) \text{ and}\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0 \text{ a.e. },\\ |h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\le 2N^2\chi_L\Longrightarrow\int|h^{(N)}_{n,L})^2-g^{(N)}_L)^2|\to 0.\end{gathered} \tag{12.30}$$

(3) Now, as $N\to\infty$, $(g^{(N)}_L)^2\to(g_L)^2$ a.e. and $(g^{(N)}_L)^2\to(g_L)^2\le f^2$ so by dominated convergence, $(g_L)^2\in\mathcal{L}^1$ and $\int|(g^{(N)}_L)^2-(g_L)^2|\to 0$ as $N\to\infty$.
(4) The same argument of dominated convergence shows now that $g^2_L\to f^2$ and $\int|g^2_L-f^2|\to 0$ using the bound by $f^2\in\mathcal{L}^1(\mathbb{R})$.

(5) What this is all for is to show that $fg \in \mathcal{L}^1(\mathbb{R})$ if f, $F = g \in \mathcal{L}^2(\mathbb{R})$ (for easier notation). Approximate each of them by sequences of step functions as above, $h_{n,L}^{(N)}$ for f and $H_{n,L}^{(N)}$ for g. Then the product sequence is in $\mathcal{L}^1$ – being a sequence of step functions – and

$$h_{n,L}^{(N)}(x)H_{n,L}^{(N)}(x) \to g_L^{(N)}(x)G_L^{(N)}(x) \tag{12.31}$$

almost everywhere and with absolute value bounded by $N^2\chi_L$. Thus by dominated convergence $g_L^{(N)}G_L^{(N)} \in \mathcal{L}^1(\mathbb{R})$. Now, let $N \to \infty$; this sequence converges almost everywhere to $g_L(x)G_L(x)$ and we have the bound

$$|g_L^{(N)}(x)G_L^{(N)}(x)| \le |f(x)F(x)|\frac{1}{2}(f^2 + F^2) \tag{12.32}$$

so as always by dominated convergence, the limit $g_L G_L \in \mathcal{L}^1$. Finally, letting $L \to \infty$ the same argument shows that $fF \in \mathcal{L}^1(\mathbb{R})$. Moreover, $|fF| \in \mathcal{L}^1(\mathbb{R})$ and

$$|\int fF| \le \int |fF| \le \|f\|_{L^2}\|F\|_{L^2} \tag{12.33}$$

where the last inequality follows from Cauchy's inequality – if you wish, first for the approximating sequences and then taking limits.

(6) So if f, $g \in \mathcal{L}^2(\mathbb{R})$ are real-value, $f + g$ is certainly locally integrable and

$$(f + g)^2 = f^2 + 2fg + g^2 \in \mathcal{L}^1(\mathbb{R}) \tag{12.34}$$

by the discussion above. For constants $f \in \mathcal{L}^2(\mathbb{R})$ implies $cf \in \mathcal{L}^2(\mathbb{R})$ is directly true.

(7) The argument is the same as for $\mathcal{L}^1$ versus L^1. Namely $\int f^2 = 0$ implies that $f^2 = 0$ almost everywhere which is equivalent to $f = 0$ a@ė. Then the norm is the same for all $f + h$ where h is a null function since fh and h^2 are null so $(f+h)^2 = f^2+2fh+h^2$. The same is true for the inner product so it follows that the quotient by null functions

$$L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N} \tag{12.35}$$

is a preHilbert space.

However, it remains to show completeness. Suppose $\{[f_n]\}$ is an absolutely summable series in $L^2(\mathbb{R})$ which means that $\sum\limits_n \|f_n\|_{L^2} < \infty$. It follows that the cut-off series $f_n\chi_L$ is absolutely summable in the L^1 sense since

$$\int |f_n\chi_L| \le L^{\frac{1}{2}}(\int f_n^2)^{\frac{1}{2}} \tag{12.36}$$

by Cauchy's inequality. Thus if we set $F_n = \sum\limits_{k-1}^{n} f_k$ then $F_n(x)\chi_L$ converges almost everywhere for each L so in fact

$$F_n(x) \to f(x) \text{ converges almost everywhere.} \tag{12.37}$$

We want to show that $f \in \mathcal{L}^2(\mathbb{R})$ where it follows already that f is locally integrable by the completeness of L^1. Now consider the series

$$g_1 = F_1^2,\ g_n = F_n^2 - F_{n-1}^2. \tag{12.38}$$

The elements are in $\mathcal{L}^1(\mathbb{R})$ and by Cauchy's inequality for $n > 1$,

(12.39)
$$\int |g_n| = \int |F_n^2 - F_{n-1}|^2 \le \|F_n - F_{n-1}\|_{L^2}\|F_n + F_{n-1}\|_{L^2} \le \|f_n\|_{L^2} 2 \sum_k \|f_k\|_{L^2}$$

where the triangle inequality has been used. Thus in fact the series g_n is absolutely summable in $\mathcal{L}^1$

$$\sum_n \int |g_n| \le 2(\sum_n \|f_n\|_{L^2})^2. \tag{12.40}$$

So indeed the sequence of partial sums, the F_n^2 converge to $f^2 \in \mathcal{L}^1(\mathbb{R})$. Thus $f \in \mathcal{L}^2(\mathbb{R})$ and moroever

$$\int (F_n - f)^2 = \int F_n^2 + \int f^2 - 2\int F_n f \to 0 \text{ as } n \to \infty. \tag{12.41}$$

Indeed the first term converges to $\int f^2$ and, by Cauchys inequality, the series of products $f_n f$ is absulutely summable in L^1 with limit f^2 so the third term converges to $-2\int f^2$. Thus in fact $[F_n] \to [f]$ in $L^2(\mathbb{R})$ and we have proved completeness.

(8) For the complex case we need to check linearity, assuming f is locally integrable and $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. The real part of f is locally integrable and the approximation $F_L^{(N)}$ discussed above is square integrable with $(F_L^{(N)})^2 \le |f|^2$ so by dominated convergence, letting first $N \to \infty$ and then $L \to \infty$ the real part is in $\mathcal{L}^2(\mathbb{R})$. Now linearity and completeness follow from the real case.

Problem 5.4
Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1 + j^2)|c_j|^2 < \infty \right\}. \tag{12.42}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c, d) \longmapsto \langle c, d\rangle = \sum_j (1 + j^2) c_j \overline{d_j} \tag{12.43}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{12.44}$$

Solution:

(1) The inner product is well defined since the series defining it converges absolutely by Cauchy's inequality:

(12.45)
$$\langle c, d\rangle = \sum_j (1 + j^2)^{\frac12} c_j \overline{(1 + j^2)^{\frac12} d_j},$$
$$\sum_j |(1 + j^2)^{\frac12} c_j \overline{(1 + j^2)^{\frac12} d_j}| \le (\sum_j (1 + j^2)|c_j|^2)^{\frac12} (\sum_j (1 + j^2)|d_j|^2)^{\frac12}.$$

It is sesquilinear and positive definite since

$$\|c\|_{2,1} = (\sum_j (1+j^2)|c_j|^2)^{\frac{1}{2}} \tag{12.46}$$

only vanishes if all c_j vanish. Completeness follows as for l^2 – if $c^{(n)}$ is a Cauchy sequence then each component $c_j^{(n)}$ converges, since $(1+j)^{\frac{1}{2}} c_j^{(n)}$ is Cauchy. The limits c_j define an element of $h^{2,1}$ since the sequence is bounded and

$$\sum_{j=1}^{N} (1+j^2)^{\frac{1}{2}} |c_j|^2 = \lim_{n\to\infty} \sum_{j=1}^{N} (1+j^2)|c_j^{(n)}|^2 \le A \tag{12.47}$$

where A is a bound on the norms. Then from the Cauchy condition $c^{(n)} \to c$ in $h^{2,1}$ by passing to the limit as $m \to \infty$ in $\|c^{(n)} - c^{(m)}\|_{2,1} \le \epsilon$.

(2) Clearly $h^{2,2} \subset l^2$ since for any finite N

$$\sum_{j=1}^{N} |c_j|^2 \sum_{j=1}^{N} (1+j)^2 |c_j|^2 \le \|c\|_{2,1}^2 \tag{12.48}$$

and we may pass to the limit as $N \to \infty$ to see that

$$\|c\|_{l^2} \le \|c\|_{2,1}. \tag{12.49}$$

Problem 5.5 In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{12.50}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{12.51}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{12.52}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H\ \forall\ u \in H \text{ and } \|T\| = \|w\|_H. \tag{12.53}$$

Solution:

(1) The finite sum $w_N = \sum\limits_{i=1}^{N} w_i e_i$ is an element of the Hilbert space with norm $\|w_N\|_N^2 = \sum\limits_{i=1}^{N} |w_i|^2$ by Bessel's identity. Expanding out

$$T(w_N) = T(\sum_{i=1}^{N} w_i e_i) = \sum_{i=1}^{n} w_i T(e_i) = \sum_{i=1}^{N} |w_i|^2 \tag{12.54}$$

and from the continuity of T,

$$|T(w_N)| \le C\|w_N\|_H \Longrightarrow \|w_N\|_H^2 \le C\|w_N\|_H \Longrightarrow \|w_N\|^2 \le C^2 \tag{12.55}$$

which is the desired inequality.

(2) Letting $N \to \infty$ it follows that the infinite sum converges and

$$\sum_i |w_i|^2 \le C^2 \Longrightarrow w = \sum_i w_i e_i \in H \tag{12.56}$$

since $\|w_N - w\| \le \sum_{j>N} |w_i|^2$ tends to zero with N.

(3) For any $u \in H$ $u_N = \sum_{i=1}^{N} \langle u, e_i \rangle e_i$ by the completness of the $\{e_i\}$ so from the continuity of T

$$\begin{aligned} T(u) = \lim_{N\to\infty} T(u_N) &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, e_i \rangle T(e_i) \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, w_i e_i \rangle = \lim_{N\to\infty} \langle u, w_N \rangle = \langle u, w \rangle \end{aligned} \tag{12.57}$$

where the continuity of the inner product has been used. From this and Cauchy's inequality it follows that $\|T\| = \sup_{\|u\|_H = 1} |T(u)| \le \|w\|$. The converse follows from the fact that $T(w) = \|w\|_H^2$.