18.102 Introduction to Functional Analysis
Spring 2009

Lecture 13. Thursday, Mar 19: Baire's theorem

Note from Lecture 9, modified and considerably extended.

Theorem 8 (Baire)**.** *If M is a non-empty complete metric space and $C_n \subset M$, $n \in \mathbb{N}$, are closed subsets such that*

$$M = \bigcup_n C_n \tag{13.1}$$

then at least one of the C_n's has an interior point.

Proof. So, choose $p_1 \notin C_1$, which must exist since otherwise C_1 contains an open ball. Since C_1 is closed there exists $\epsilon_1 > 0$ such that $B(p_1, \epsilon_1) \cap C_1 = \emptyset$. Next choose $p_2 \in B(p_1, \epsilon_1/3)$ which is not in C_2, which is possible since otherwise $B(p_1, \epsilon_1/3) \subset C_2$, and $\epsilon_2 > 0$, $\epsilon_2 < \epsilon_1/3$ such that $B(p_2, \epsilon_2) \cap C_2 = \emptyset$. So we have used both the fact that C_2 has empty interior and the fact that it is closed. Now, proceed, inductively. Assume that there is a finite sequence p_i, $i = 1, \dots, k$ and positive numbers $0 < \epsilon_k < \epsilon_{k-1}/3 < \epsilon_{k-2}/3^2 < \cdots < \epsilon_1/3^{k-1} < 3^{-k}$ such that $p_j \in B(p_{j-1}, \epsilon_{j-1}/3)$ and $B(p_j, \epsilon_j) \cap C_j = \emptyset$. Then we can add another p_{k+1} by using the properties of C_k – it has non-empty interior so there is some point in $B(p_k, \epsilon_k/3)$ which is not in C_{k+1} and then $B(p_{k+1}, \epsilon_{k+1}) \cap C_{k+1} = \emptyset$ where $\epsilon_{k+1} > 0$ but $\epsilon_{k+1} < \epsilon_k/3$. Thus, we have constructe and infinite sequence $\{p_k\}$ in M. Since $d(p_{k+1}, p_k) < \epsilon_k/3$ this is a Cauchy sequence. In fact

$$d(p_k, p_{k+l}) < \epsilon_k/3 + \cdots + \epsilon_{k+l-1}/3 < 3^{-k} < 2\epsilon_k/3 \tag{13.2}$$

for all $l > 0$, and this tends to zero as $k \to \infty$.

Since M is complete this sequence converges. From (13.2) the limit, $q \in M$ must lie in the closure of $B(p_k, 2\epsilon_k/3)$ for every k. Hence $q \notin C_k$ for any k which contradicts (13.1).

Thus, at least one of the C_n must have non-empty interior. □

One application of this is often called the *uniform boundedness principle,* I will just call it:

Theorem 9 (Uniform boundedness)**.** *Let B be a Banach space and suppose that T_n is a sequence of bounded (i.e. continuous) linear operators $T_n : B \longrightarrow V$ where V is a normed space. Suppose that for each $b \in B$ the set $\{T_n(b)\} \subset V$ is bounded (in norm of course) then* $\sup_n \|T_n\| < \infty$.

Proof. This follows from a pretty direct application of Baire's theorem to B. Consider the sets

$$S_p = \{b \in B,\ \|b\| \le 1,\ \|T_n b\|_V \le p\ \forall\ n\},\ p \in \mathbb{N}. \tag{13.3}$$

Each S_p is closed because T_n is continuous, so if $b_k \to b$ is a convergent sequence then $\|b\| \le 1$ and $\|T_n(p)\| \le p$. The union if the S_p is the whole of the closed ball of radius one around the origin in B :

$$\{b \in B; d(b, 0) \le 1\} = \bigcup_p S_p \tag{13.4}$$

because of the assumption of 'pointwise boundedness' – each b with $\|b\| \le 1$ must be in one of the S_p's.

So, by Baire's theorem one of the sets S_p has non-empty interior. This means that for some p, some $v \in S_p$, and some $\delta > 0$,

$$w \in B,\ \|w\|_B \le \delta \Longrightarrow \|T_n(v+w)\|_V \le p\ \forall\ n. \tag{13.5}$$

Moving v to $(1-\delta/2)v$ and halving δ as necessary it follows that this ball $B(v,\delta)$ is contained in the open ball around the origin of radius 1. Thus, using the triangle inequality, and the fact that $\|T_n(v)\|_V \le p$ this implies

$$w \in B,\ \|w\|_B \le \delta \Longrightarrow \|T_n(w)\|_V \le 2p \Longrightarrow \|T_n\| \le 2p/\delta \tag{13.6}$$

since the norm of the operator is $\sup\{\|Tw\|_V; \|w\|_B = 1\|$ it follows that the norms are uniformly bounded:

$$\|T_n\| \le 2p/\delta \tag{13.7}$$

as claimed. □

One immediate consequence of this is that, as I mentioned in last lecture, it is not necessary to assume that a weakly convergent sequence in a Hilbert space is norm bounded.

Corollary 2. *If $u_n \in H$ is a sequence in a Hilbert space and for all $v \in H$*

$$(u_n, v) \to F(v) \text{ converges in } \mathbb{C} \tag{13.8}$$

then $\|u_n\|_H$ is bounded and there exists $w \in H$ such that $u_n \rightharpoonup w$ (converges weakly).

Proof. Well, a corollary really should not need a proof but still I will give one since maybe it is a bit more than a corollary.

Apply the Uniform Boundedness Theorem to the continuous functionals

$$T_n(u) = (u, u_n),\ T_n : H \longrightarrow \mathbb{C} \tag{13.9}$$

where we reverse the order to make them linear rather than anti-linear. Thus, each set $|T_n(u)|$ is bounded in $\mathbb{C}$ since it is convergent. It follows that there is a bound

$$\|T_n\| \le C. \tag{13.10}$$

However, the norm is just $\|T_n\| = \|u_n\|_H$ so the sequence must be bounded in H. Define $T : H \longrightarrow \mathbb{C}$ as the limit for each u :

$$T(u) = \lim_{n\to\infty} T_n(u) = \lim_{n\to\infty} (u, u_n). \tag{13.11}$$

This exists for each u by hypothesis. It is a linear map an from (13.10) it is bounded, $\|T\| \le C$. Thus by the Riesz Representation theorem, there exists $w \in H$ such that

$$T(u) = (u, w)\ \forall\ u \in H. \tag{13.12}$$

Thus $(u_n, u) \to (w, u)$ for all $u \in H$ so $u_n \rightharpoonup w$ as claimed. □

The second major application of Baire's theorem is to

Theorem 10 (Open Mapping)**.** *If $T : B_1 \longrightarrow B_2$ is a bounded and surjective linear map between two Banach spaces then T is open:*

$$T(O) \subset B_2 \text{ is open if } O \subset B_1 \text{ is open.} \tag{13.13}$$

This is 'wrong way continuity' and as such can be used to prove the continuity of inverse maps as we shall see. The proof uses Baire's theorem but then another similar sort of argument is needed. I did not finish the second argument in the lecture.

Proof. (1) The first part, of the proof, using Baire's theorem shows that the *closure* of the image, so in B_2, of an open ball around the origin in B_1, has 0 as an interior point – i.e. it contains an open ball around the origin in B_2. To see this we apply Baire's theorem to the sets

$$C_p = \mathrm{cl}_{B_2} T(B(0,p)) \tag{13.14}$$

the closure of the image of the ball in B_1 or radius p. We need to take the closure since the sets in Baire's theorem are closed, but even before doing that we know that

$$B_2 = \sum_p T(B(0,p)) \tag{13.15}$$

since that is what surjectivity means – every point is the image of something. Thus one of the closed sets C_p has an interior point, v. Since T is surjective, $v = Tu$ for some $u \in B_1$. The sets C_p increase with p so we can take a larger p and v is still an interior point, from which it follows that $0 = v - Tu$ is an interior point as well. Thus indeed

$$C_p \supset B(0,\delta) \tag{13.16}$$

for some $\delta > 0$.

(2) Having applied Baire's thereom, consider now what (13.16) means. It follows that each $v \in B_2$ with $\|v\| < \delta$ is the limit of a sequence Tu_n where $\|u_n\| \le p$. What we want to arrange is that this sequence converges. Note that we can scale the norm of v using the linearity of T. Thus, for a general $v \in B_2$ we can apply (13.16) to $v' = \delta v/2\|v\|$ to see that $Tu'_n \to v'$ where $\|u'_n\| \le p$. Then $u_n = \|v\|u'_n/\delta$ satisfies $Tu_n \to v$, $\|u_n\| \le 2p\|v\|/\delta$. To simplifiy the arithmetic, let me replace T by cT where $c = p/2\delta$. This means that for each $v \in B_2$ there is a sequence u_n in B_1 with $\|u_n\| \le \|v\|$ and $Tu_n \to v$.

Now, we can stop before we get to the limit of the sequence and get as close to v as we want. This means that

$$\text{For each } v \in B_2,\ \exists\ u \in B_1,\ \|u\| < \|v\|,\ \|v - Tu\| \le \frac{1}{2}\|v\|. \tag{13.17}$$

This in turn we can iterate to construct a *better* sequence. Fix $w = w_1 \in B_1$ with $\|w_1\| < 1$ and choose $u_1 = u$ according to (13.17) for $v = w = w_1$. Thus $\|u_1\| < 1$ and $w_2 = w_1 - Tu_1$ satisfies $\|w_2\| < \frac{1}{2}$. Now proceed by induction, supposing that we have constructed a sequence u_j, $j < n$, in B_1 with $\|u_j\| \le 2^{-j+1}$ and $\|w_{j1}\| < 2^{-j+1}$ where $w_j = w_{j-1} - Tu_{j-1}$. Then we can choose u_n to extend the induction and so we get a sequence u_n such that for each n

$$w - T(\sum_{j=1}^{n}) = w_1 - Tu_1 - \sum_{j=2}^{n} = w_2 - Tu_2 - \sum_{j=3}^{n} = w_{n+1}. \tag{13.18}$$

The *series* with terms u_n is absolutely summabel, hence convergent since B_1 is complete, and

$$w = Tu,\ u = \sum_j u_j,\ \|u\| \le 2. \tag{13.19}$$

So finally we have shown that each $w \in B(0,1)$ in B_2 is in the image of $B(0,2)$ in B_1. Going back to the unscaled T it follows that for some $\delta > 0$,

$$B(0,\delta) \subset T(B(0,1)). \tag{13.20}$$

(3) It follows of course that the image $T(O)$ of any open set is open, since if $w \in T(O)$ then $w = Tu$ for some $u \in O$ and hence $B(w, \epsilon\delta)$ is contained in the image of $u + B(0,\epsilon) \subset O$ for $\epsilon > 0$ sufficiently small.

□

So, as I did not quite finish the proof in lecture. However at the very end I mention the two most important applications of of this 'Open Mapping Theorem'. Namely:

Corollary 3. *If $T : B_1 \longrightarrow B_2$ is a bounded linear map between Banach spaces which is 1-1 and onto, i.e. is a bijection, then it is a homeomorphism – meaning its inverse, which is necessarily linear, is also bounded.*

Proof. The only confusing thing is the notation. Note that T^{-1} is used to denote the inverse maps on sets. So, the inverse of T, let's call it $S : B_2 \longrightarrow B_2$ is certainly linear. If $O \subset B_1$ is open then $S^{-1}(O) = T(O)$ is open by the Open Mapping theorem, so S is continuous. □

The second application is

Theorem 11 (Closed Graph)**.** *If $T : B_1 \longrightarrow B_2$ is a linear map between Banach spaces then it is bounded if and only if its graph*

$$\mathrm{Gr}(T) = \{(u,v) \in B_1 \times B_2; u_2 = Tu_1\} \tag{13.21}$$

is a closed subset of the Banach space $B_1 \times B_2$.

Have we actually covered the product of Banach spaces explicitly? If not, think about it for a minute or two!

Proof. Suppose first that T is bounded, i.e. continuous. A sequence $(u_n, v_n) \in B_1 \times B_2$ is in $\mathrm{Gr}(T)$ if and only if $v_n = Tu_n$. So, if it converges, then $u_n \to u$ and $v_n = Tu_n \to Tv$ by the continuity of T, so the limit is in $\mathrm{Gr}(T)$ which is therefore closed.

Conversely, suppose the graph is closed. Given the graph we can reconstruct the map it comes from (whether linear or not) from a little diagram. Form $B_1 \times B_2$ consider the two projections, $\pi_1(u,v) = u$ and $\pi_2(u,v) = v$. Both of them are continuous by inspection and we can restrict them to $\mathrm{Gr}(T) \subset B_1 \times B_2$ to get

$$\begin{tikzcd} & \mathrm{Gr}(T) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B_1 \arrow[rr, "T"] & & B_2. \end{tikzcd} \tag{13.22}$$

This little diagram commutes. Indeed there are two ways to map a point $(u,v) \in \mathrm{Gr}(T)$ to B_2, either directly, sending it to v or first sending it $u \in B_1$ and then to Tu. Since $v = Tu$ these are the same.

Now, $\mathrm{Gr}(T) \subset B_1 \times B_2$ is a closed subspace, so it too is a Banach space and π_1 and π_2 remain continuous when restricted to it. The map π_1 is 1-1 and onto, because each u occurs as the first element of precisely one pair, namely $(u, Tu) \in \mathrm{Gr}(T)$.

Thus the Corollary above applies to π_1 to show that its inverse, S is continuous. But then $T = \pi_2 \circ S$, from the commutativity, is also continuous proving the theorem. $\square$