18.102 Introduction to Functional Analysis
Spring 2009

Lecture 15. Thursday, April 2

I recalled the basic properties of the Banach space, and algebra, of bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. In particular that it is a Banach space with respect to the norm

$$\|A\| = \sup_{\|u\|_{\mathcal{H}}=1} \|Au\|_{\mathcal{H}} \tag{15.1}$$

and that the norm satisfies

$$\|AB\| \le \|A\|\|b\|. \tag{15.2}$$

Restatated and went through the proof again of the

Theorem 13 (Open Mapping)**.** *If $A : B_1 \longrightarrow B_2$ is a bounded linear operator between Banach spaces and $A(B_1) = B_2$, i.e. A is surjective, then it is open:*

$$A(O) \subset B_2 \textit{ is open } \forall\ O \subset B_1 \textit{ open.} \tag{15.3}$$

Proof in Lecture 13, also the two consequences of it: If $A : B_1 \longrightarrow B_2$ is bounded, 1-1 and onto (so it is a bijection) then its inverse is also bounded. Secondly the closed graph theorem. All this is in the notes for Lecture 13.

As a second example of the Uniform Boundedness Theorem I also talked about strong convergence of operators. Thus a sequence of bounded operators (on a separable Hilbert space) $A_n \in \mathcal{B}(\mathcal{H})$ is said to *converge strongly* if for each $u \in \mathcal{H}$ $A_n u$ converges. It follows that the limit is a bounded linear operator – or you can include this in the definition if you prefer. The Uniform Boundedness Theorem shows that if A_n is strongly convergent then it is bounded, $\sup_n \|A_n\| < \infty$. You will need this for the problems this week.

I also talked about the shift operator $S : l^2 \longrightarrow l^2$ defined by

$$S(\sum_{j=1}^{\infty} c_j e_j) = \sum_{j=1}^{\infty} c_j e_{j+1} \tag{15.4}$$

defined by moving each element of the sequence 'up one' and starting with zero. This is an example of a bounded linear operator, with $\|S\| = 1$ clearly enough, which is 1-1, since $Au = 0$ implies $u = 0$, but which is not surjective. Indeed the range of S is exactly the subspace

$$H_1 = \{u \in L^2; (u, e_1) = 0\}. \tag{15.5}$$

Using the open mapping theorem (or directly) it is easy to see that S is invertible as a bounded linear map from H to H_1, but not on H. In fact as you should show in the problem set this week, it cannot be made invertible by a small perturbation. This shows in particular that the set of invertible elements of $\mathcal{B}(\mathcal{H})$ is not dense, which is quite different from the finite dimensional case.

Finally I started to talk about the set of invertible elements:

$$\mathrm{GL}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}); \exists\ B \in \mathcal{H}(\mathcal{H}),\ BA = AB = \mathrm{Id}\}. \tag{15.6}$$

Note that this is equivalent to saying A is 1-1 and onto in view of the discussion above.

Lemma 10. *If $A \in \mathcal{B}(\mathcal{H})$ and $\|A\| < 1$ then*

$$\mathrm{Id} - A \in \mathrm{GL}(\mathcal{H}). \tag{15.7}$$

Proof. Neumann series. If $\|A\| < 1$ then $\|A^j\| \le \|A\|^j$ and it follows that the Neumann series

$$B = \sum_j A^j \tag{15.8}$$

is absolutely summable in $\mathcal{B}(\mathcal{H})$ sicne $\sum\limits_{j=0}^{1} \|A^j\|$ converges. Thus the sum converges. Moreover by the continuity of the product with respect to the norm

$$AB = A \lim_{n\to\infty} \sum_{j=0}^{n} A^j = \lim_{n\to\infty} \sum_{j=1}^{n+1} A^j = B - \mathrm{Id} \tag{15.9}$$

an similarly $BA = B - \mathrm{Id}$. Thus $(\mathrm{Id} - A)B = B(\mathrm{Id} - A) = \mathrm{Id}$ shows that B is a (and hence the) 2-sided inverse of $\mathrm{Id} - A$. □

Proposition 22. *The group of invertible elements* $\mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ *is open (but not dense if* $\mathcal{H}$ *is infinite-dimensional).*

Proof. I will do the proof next time. □