18.102 Introduction to Functional Analysis
Spring 2009

Lecture 18. Tuesday April 14: Compact operators

Last time we considered invertible elemenets of $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a separable Hilbert space, and also the finite rank operators. The latter form an ideal which is closed under taking adjoints. We also showed the the closure of this ideal, the elements in $\mathcal{B}(\mathcal{H})$ which are the limits of (norm-convergent) sequences of finite rank operators, also form an ideal which is closed under taking adjoints and also norm, i.e. metrically, closed.

Definition 8. An element $K \in \mathcal{B}(\mathcal{H})$, the bounded operators on a separable Hilbert space, is said to be *compact* (the old terminology was 'totally bounded' and you might still see this) if the image of the unit ball is precompact, i.e. has compact closure – that is if the closure of $K\{u \in \mathcal{H}; \|u\|_{\mathcal{H}} \leq 1\}$ is compact in $\mathcal{H}$.

Lemma 12. *An operator $K \in \mathcal{B}(\mathcal{H})$ is compact if and only if the image $\{Ku_n\}$ of any weakly convergent sequence $\{u_n\}$ in $\mathcal{H}$ is strongly, ie. norm, convergent.*

Proof. First suppose that $u_n \rightharpoonup u$ is a weakly convergent sequence in $\mathcal{H}$ and that K is compact. We know that $\|u_n\| < C$ is bounded so the sequence Ku_n is contained in $CK(B(0,1))$ and hence in a compact set (clearly if D is compact then so is cD for any constant c.) Thus, any subsequence of Ku_n has a convergent subseqeunce and the limit is necessarily Ku since $Ku_n \rightharpoonup Ku$ (true for any bounded operator by computing

$$(Ku_n, v) = (u_n, K^*v) \to (u, K^*v) = (Ku, v).) \tag{18.1}$$

But the condition on a sequence in a metric space that every subsequence of it has a subsequence which converges to a fixed limit implies convergence. (If you don't remember this, reconstruct the proof: To say a sequence v_n *does not* converge to v is to say that for some $\epsilon > 0$ there is a subsequence along which $d(v_{n_k}, v) \geq \epsilon$. This is impossible given the subsequence of subsequence condition (coverging to the fixed limit v.)

Conversely, suppose that K has this property of turning weakly convergent into strongly convergent sequences. We want to show that $K(B(0,1))$ has compact closure. This just means that any sequence in $K(B(0,1))$ has a (strongly) convergent subsequence – where we do not have to worry about whether the limit is in the set or not. Such a sequence is of the form Ku_n where u_n is a sequence in $B(0,1)$. However we know that the ball is weakly compact, that is we can pass to a subsequence which converges weakly, $u_{n_j} \rightharpoonup u$. Then, by the assumption of the Lemma, $Ku_{n_j} \to Ku$ converges strongly. Thus u_n does indeed have a convergent subsequence and hence $K(B(0,1))$ must have compact closure. □

Proposition 25. *An operator $K \in \mathcal{B}(\mathcal{H})$, bounded on a separable Hilbert space, is compact if and only if it is the limit of a norm-convergent sequence of finite rank operators, i.e. the ideal of compact operators $\mathcal{K}(\mathcal{H})$ is the norm closure of the ideal of finite rank operators.*

Proof. So, we need to show that a compact operators is the limit of a convergent sequence of finite rank operators. To do this we use one of the characterizations of compact subsets of a separable Hilbert space discussed earlier. Namely, if e_i is an orthonormal basis of $\mathcal{H}$ then a subset $I \subset \mathcal{H}$ is compact if and only if it is closed and bounded and has equi-small tails with respec to $\{e_i\}$, meaning given $\epsilon > 0$

there exits N such that

$$\sum_{i>N} |(v, e_i)|^2 < \epsilon^2 \ \forall\ v \in I. \tag{18.2}$$

Now we shall apply this to the set $K(B(0,1))$ where we assume that K is compact – so this set is *contained* in a compact set. Hence (18.2) applies to it. Namely this means that for any $\epsilon > 0$ there exists n such that

$$\sum_{i>n} |(Ku, e_i)|^2 < \epsilon^2 \ \forall\ u \in \mathcal{H},\ \|u\|_{\mathcal{H}} \le 1. \tag{18.3}$$

For each n consider the first part of these sequences and define

$$K_n u = \sum_{k \le n} (Ku, e_i) e_i. \tag{18.4}$$

This is clearly a linear operator and has finite rank – since its range is contained in the span of the first n elements of $\{e_i\}$. Since this is an orthonormal basis,

$$\|Ku - K_n u\|_{\mathcal{H}}^2 = \sum_{i>n} |(Ku, e_i)|^2 \tag{18.5}$$

Thus (18.3) shows that $\|Ku - K_n u\|_{\mathcal{H}} \le \epsilon$. Now, increasing n makes $\|Ku - K_n u\|$ smaller, so given $\epsilon > 0$ there exits n such that for all $N \ge n$,

$$\|K - K_N\|_{\mathcal{B}} = \sup_{\|u\| \le 1} \|Ku - K_n u\|_{\mathcal{H}} \le \epsilon. \tag{18.6}$$

Thus indeed, $K_n \to K$ in norm and we have shown that the compact operators are contained in the norm closure of the finite rank operators.

For the converse we assume that $T_n \to K$ is a norm convergent sequence in $\mathcal{B}(\mathcal{H})$ where each of the T_n is of finite rank – of course we know nothing about the rank except that it is finite. We want to conclude that K is compact, so we need to show that $K(B(0,1))$ is precompact. It is certainly bounded, by the norm of K. By one of the results on compactness of sets in a separable Hilbert space we know that it suffices to prove that every weakly convergent sequence in $K(B(0,1))$ has a strongly convergent subsequence – meaning norm convergent. The limit need not be in $K(B(0,1))$ but must exist of the set is to have compact closure. So, suppose v_k is a weakly convergent sequence in $K(B(0,1))$. Well then, it is of the form Ku_k where u_k is a sequence in the unit ball. Of necessity this has a weakly convergent subsequence, so we can assume that $u_k \rightharpoonup u$ is weakly convergent, by passing to a subsequence of the original sequence. Now, each T_n is of finite rank so the sequences $T_n v_k$ are each strongly convergent as $k \to \infty$ – namely they are weakly convergent because $(T_n v_k, w) = (v_k, T_n^* w)$, and in a finite dimensional space. Use the triangle inequality and definition of the norm of an operator to see that

$$\begin{aligned} \|Kv_k - Kv_l\| \le \|Kv_k - T_n v_k\| + \|T_n v_k - T_n v_l\| + \|T_n v_l - Kv_l\| \\ \le 2\|K - T_n\|_{\mathcal{B}} + \|T_n v_k - T_n v_l\|. \end{aligned} \tag{18.7}$$

Now, given $\epsilon > 0$ first choose n so large that $\|K - T_n\| < \epsilon/3$. Then, having fixed n, use the fact that $T_n v_k$ is Cauchy to choose p such that $k, l > p$ implies $\|T_n v_k - T_n v_l\| < \epsilon/3$. It follows that Kv_k is Cauchy and hence convergent by the completeness of Hilbert space. Thus K is compact. □

Notice that this shows that the ideal of compact operators is itself closed – you can see this from the last argument but of course it follows from the fact that it is the closure of the finite rank operators.

Solutions to Problem set 8

Problem 8.1 Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{18.8}$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

Problem 8.2

Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{18.9}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

Problem 8.3 Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx + ily)/2\pi$, $k, l \in \mathbb{Z}$, form a complete orthonormal basis.