18.102 Introduction to Functional Analysis
Spring 2009

Lecture 22. Thursday April 30: Dirchlet problem continued

I did not finish the proof last time:-

Proof. Notice the form of the solution in case $V \geq 0$ in (21.25). In general, we can choose a constant c such that $V + c \geq 0$. Then the equation

$$-\frac{d^2w}{dx^2} + Vw = Tw_k \Longleftrightarrow -\frac{d^2w}{dx^2} + (V+c)w = (T+c)w. \tag{22.1}$$

Thus, if w satisfies this eigen-equation then it also satisfies

$$w = (T+c)A(\operatorname{Id} + A(V+c)A)^{-1}Aw \Longleftrightarrow \\ Sw = (T+c)^{-1}w,\ S = A(\operatorname{Id} + A(V+c)A)^{-1}A. \tag{22.2}$$

Now, we have shown that S is a compact self-adjoint operator on $L^2(0, 2\pi)$ so we know that it has a complete set of eigenfunctions, e_k, with eigenvalues $\tau_k \neq 0$. From the discussion above we then know that each e_k is actually continuous – since it is Aw' with $w' \in L^2(0, 2\pi)$ and hence also twice continuously differentiable. So indeed, these e_k satisfy the eigenvalue problem (with Dirichlet boundary conditions) with eigenvalues

$$T_k = \tau_k^{-1} + c \to \infty \text{ as } k \to \infty. \tag{22.3}$$

The solvability part also follows much the same way. □