18.102 Introduction to Functional Analysis
Spring 2009

## Lecture 23. Tuesday, May 5: Harmonic oscillator

As a second 'serious' application of at least the completeness part of the spectral theorem for self-adjoint compact operators, I want to discuss the Hermite basis for $L^2(\mathbb{R})$. Note that so far we have not found an explicit orthonormal basis on the whole real line, even though we know $L^2(\mathbb{R})$ to be separable, so we certainly know that such a basis exists. How to construct one explicitly and with some handy properties? One way is to simply orthonormalize – using Gramm-Schmidt – some countable set with dense span. For instance consider the basic Gaussian function

$$\exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}). \tag{23.1}$$

This is so rapidly decreasing at infinity that the product with any polynomial is also square integrable:

$$x^k \exp(-\frac{x^2}{2}) \in L^2(\mathbb{R}) \ \forall \ k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}. \tag{23.2}$$

Orthonormalizing this sequence gives an orthonormal basis, where completeness can be shown by an appropriate approximation technique.

Rather than proceed directly we will discuss the invertibility of the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2 \tag{23.3}$$

which we want to think of as an operator – although for the moment I will leave vague the question of what it operates *on*.

The first thing to observe is that the Gaussian is an eigenfunction of $H$

$$He^{-x^2/2} = -\frac{d}{dx}(-xe^{-x^2/2} + x^2e^{-x^2/2} \\ - (x^2-1)e^{-x^2/2} + x^2e^{-x^2/2} = e^{-x^2/2} \tag{23.4}$$

with eigenvalue 1 – for the moment this is only in a formal sense.

In this special case there is an essentially algebraic way to generate a whole sequence of eigenfunctions from the Gaussian. To do this, write

$$Hu = (-\frac{d}{dx} + x)(\frac{d}{dx} + x)u + u = (CA + 1)u, \\ C = (-\frac{d}{dx} + x),\ A = (\frac{d}{dx} + x) \tag{23.5}$$

again formally as operators. Then note that

$$Ae^{-x^2/2} = 0 \tag{23.6}$$

which again proves (23.4). The two operators in (23.5) are the 'creation' operator and the 'annihilation' operator. They almost commute in the sense that

$$(AC - CA)u = 2u \tag{23.7}$$

for say any twice continuously differentiable function $u$.

Now, set $u_0 = e^{-x^2/2}$ which is the 'ground state' and consider $u_1 = Cu_0$. From (23.7), (23.6) and (23.5),

$$Hu_1 = (CAC + C)u_0 = C^2Au_0 + 3Cu_0 = 3u_1. \tag{23.8}$$

Thus, $u_1$ is an eigenfunction with eigenvalue 3.

**Lemma 18.** *For $j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ the function $u_j = C^j u_0$ satisfies $Hu_j = (2j+1)u_j$.*

*Proof.* This follows by induction on $j$, where we know the result for $j = 0$ and $j = 1$. Then

$$HCu_j = (CA + 1)Cu_j = C(H - 1)u_j + 3Cu_j = (2j + 3)u_j. \tag{23.9}$$

□

Again by induction we can check that $u_j = (2^j x^j + q_j(x))e^{-x^2/2}$ where $q_j$ is a polynomial of degree at most $j - 2$. Indeed this is true for $j = 0$ and $j = 1$ (where $q_0 = q_1 \equiv 0$) and then

$$Cu_j = (2^{j+1}x^{j+1} + Cq_j)e^{-x^2/2}. \tag{23.10}$$

and $q_{j+1} = Cq_j$ is a polynomial of degree at most $j - 1$ – one degree higher than $q_j$.

From this it follows in fact that the finite span of the $u_j$ consists of all the products $p(x)e^{-x^2/2}$ where $p(x)$ is any polynomial.

Now, all these functions are in $L^2(\mathbb{R})$ and we want to compute their norms. First, a standard integral computation[3] shows that

$$\int_{\mathbb{R}} (e^{-x^2/2})^2 = \int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi} \tag{23.11}$$

For $j > 0$, integration by parts (easily justified by taking the integral over $[-R, R]$ and then letting $R \to \infty$) gives

$$\int_{\mathbb{R}} (C^j u_0)^2 = \int_{\mathbb{R}} C^j u_0(x) C^j u_0(x) dx = \int_{\mathbb{R}} u_0 A^j C^j u_0. \tag{23.12}$$

Now, from (23.7), we can move one factor of $A$ through the $j$ factors of $C$ until it emerges and 'kills' $u_0$

$$AC^j u_0 = 2C^{j-1}u_+ CAC^{j-1}u_0 = 2C^{j-1}u_0 + C^2AC^{j-2}u_0 = 2jC^{j-1}u_0. \tag{23.13}$$

So in fact,

$$\int_{\mathbb{R}} (C^j u_0)^2 = 2j \int_{\mathbb{R}} (C^{j-1}u_0)^2 = 2^j j! \sqrt{\pi}. \tag{23.14}$$

A similar argument shows that

$$\int_{\mathbb{R}} u_k u_j = \int_{\mathbb{R}} u_0 A^k C^j u_0 = 0 \text{ if } k \neq j. \tag{23.15}$$

Thus the functions

$$e_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}C^j e^{-x^2/2} \tag{23.16}$$

form an orthonormal sequence in $L^2(\mathbb{R})$.

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[3]To compute the Gaussian integral, square it and write as a double integral then introduct polar coordinates

$$\left(\int_{\mathbb{R}} e^{-x^2}dx\right)^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta = \pi\big[-e^{-r^2}\big]_0^\infty = \pi.$$

We would like to show this is orthonormal sequence is complete. Rather than argue through approximation, we can guess that in some sense the operator

$$AC = (\frac{d}{dx} + x)(-\frac{d}{dx} + x) = -\frac{d^2}{dx^2} + x^2 + 1 \tag{23.17}$$

should be invertible, so one approach ist to try to construct its 'inverse' and show this really is a compact, self-adjoint operator on $L^2(\mathbb{R})$ and that its only eigenfunctions are the $e_i$ in (23.16). Rather than do this I will proceed more indirectly.

## Solutions to Problem set 10

*Problem P10.1* Let $H$ be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of $H$ is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{23.18}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H,\ \text{1-1 and onto},\ \|u\|_H = \|Tu\|_{H\oplus H} \tag{23.19}$$

or otherwise. In any case, construct a map as in (23.19).

Solution: Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $H$, which exists by virtue of the fact that it is an infinite-dimensional but separable Hilbert space. Define the map

$$T : H \ni u \longrightarrow (\sum_{i=1}^{\infty}(u, e_{2i-1})e_i, \sum_{i=1}^{\infty}(u, e_{2i})e_i) \in H \oplus H \tag{23.20}$$

The convergence of the Fourier Bessel series shows that this map is well-defined and linear. Injectivity similarly follows from the fact that $Tu = 0$ in the image implies that $(u, e_i) = 0$ for all $i$ and hence $u = 0$. Surjectivity is also clear from the fact that

$$S : H \oplus H \ni (u_1, u_2) \longmapsto \sum_{i=1}^{\infty}((u_1, e_i)e_{2i-1} + (u_2, e_i)e_{2i}) \in H \tag{23.21}$$

is a 2-sided inverse and Bessel's identity implies isometry since $\|S(u_1, u_2)\|^2 = \|u_1\|^2 + \|u_2\|^2$

*Problem P10.2* One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if $H$ is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l_2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\} \tag{23.22}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to $H$.

Solution: A similar argument as in the previous problem works. Take an orthormal basis $e_i$ for $H$. Then the elements $E_{i,j} \in l_2(H)$, which for each $i, i$ consist of the sequences with 0 entries except the $j$th, which is $e_i$, given an orthonromal basis for $l_2(H)$. Orthormality is clear, since with the inner product is

$$(u, v)_{l_2(H)} = \sum_j (u_j, v_j)_H. \tag{23.23}$$

Completeness follows from completeness of the orthonormal basis of $H$ since if $v = \{v_j\}$ $(v, E_{j,i}) = 0$ for all $j$ implies $v_j = 0$ in $H$. Now, to construct an isometric isomorphism just choose an isomorphism $m : \mathbb{N}^2 \longrightarrow \mathbb{N}$ then

$$Tu = v,\ v_j = \sum_i (u, e_{m(i,j)})e_i \in H. \tag{23.24}$$

I would expect you to go through the argument to check injectivity, surjectivity and that the map is isometric.

*Problem P10.3* Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[4] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi i b) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi i F(q)) = f(q), \ \forall \ q \in Q \text{ and } F(0) = b. \tag{23.25}$$

Of course, you are free to change $b$ to $b + n$ for any $n \in \mathbb{Z}$ but then $F$ changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of $F$ for $N = 1$ and $f = c$. Show that the winding number of the closed curve $c$ may be defined *unambiguously* as

$$\mathrm{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{23.26}$$

Solution: Let $C'$, be another choice of $F$ in this case. Now, $g(t) = C'(t) - C(t)$ is continuous and satisfies $\exp(2\pi g(t)) = 1$ for all $t \in [0,1]$ so by the uniqueness must be constant, thus $C'(1) - C'(0) = C(1) - C(0)$ and the winding number is well-defined.

(2) Show that $\mathrm{wn}(c)$ is *constant under homotopy.* That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0, x) = c_1(x)$, $f(1, x) = c_2(x)$ for all $x \in [0,1]$ and $f(y, 0) = f(y, 1)$ for all $y \in [0,1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

Solution: Choose $F$ using the 'fact' corresponding to this homotopy $f$. Since $f$ is periodic in the second variable – the two curves $f(y, 0)$, and $f(y, 1)$ are the same – so by the uniquess $F(y, 0) - F(y, 1)$ must be constant, hence $\mathrm{wn}(c_2) = F(1, 1) - F(1, 0) = F(0, 1) - F(0, 0) = \mathrm{wn}(c_1)$.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi i x} \operatorname{Id}_{n\times n}$ of $n \times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n \times n$ matrices, such that $G(0, x) = L_n(x)$, $G(1, x) \equiv \operatorname{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y, 0) = G(y, 1)$ for all $y \in [0,1]$.

Solution: The determinant is a continuous (actually it is analytic) map which vanishes precisely on non-invertible matrices. Moreover, it is given by the product of the eigenvalues so

$$\det(L_n) = \exp(2\pi i x n). \tag{23.27}$$

This is a periodic curve with winding number $n$ since it has the 'lift' $xn$. Now, if there were to exist such an homotopy of periodic curves of matrices, always invertible, then by the previous result the winding number of the determinant would have to remain constant. Since the winding number for the constant curve with value the identity is 0 such an homotopy cannot exist.

*Problem P10.4* Consider the closed curve corresponding to$L_n$ above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi i x} \operatorname{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{23.28}$$

[4] Of course, you are free to give a proof – it is not hard.

taking values in the invertible operators on $H$. Show that after identifying $H$ with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{23.29}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi ix}u_1,u_2),\ M(y,0) = M(y,1),\ \forall\, x,y \in [0,1]. \tag{23.30}$$

Hint: So, think of $H \oplus H$ as being 2-vectors $(u_1,u_2)$ with entries in $H$. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{23.31}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H \oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2,-u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{23.32}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi ix)$ on the first and the second component respectively, leaving the other fixed.

Solution: Certainly $U(y)$ is invertible since its inverse is $U(-y)$ as follows in the two dimensional case. Thus the map $W(x,y)$ on $[0,1]^2$ in (23.32) consists of invertible and bounded operators on $H \oplus H$, meaning a continuous map $W : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H)$. When $x = 0$ or $x = 1$, both $V_1(x)$ and $v_2(x)$ reduce to the identiy, and hence $W(0,y) = W(1,y)$ for all $y$, so $W$ is periodic in $x$. Moreove at $y = 0$ $W(x,0) = V_2(x)V_1(x)$ is exactly $L(x)$, a multiple of the identity. On the other hand, at $x = 1$ we can track composite as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{2\pi ix}u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{2\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ -e^{4\pi x}u_1 \end{pmatrix} \longmapsto \begin{pmatrix} e^{4\pi x}u_1 \\ u_2 \end{pmatrix}. \tag{23.33}$$

This is what is required of $M$ in (23.30).

*Problem P10.5* Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{23.34}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi ix}u_1, e^{-2\pi ix}u_2), \\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{23.35}$$

Solution: We can take

$$G(y,x) = U(-y)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & e^{-2\pi ix} \end{pmatrix} U(y) \begin{pmatrix} e^{2\pi ix} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{23.36}$$

By the same reasoning as above, this is an homotopy of closed curves of invertible operators on $H \oplus H$ which satisfies (23.35).

*Problem P10.6* Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (23.34), $\tilde{G} : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde{G}(0,x)\,\{u_k\}_{k=1}^{\infty} &= \left\{\exp((-1)^k 2\pi ix)u_k\right\}_{k=1}^{\infty}, \\ \tilde{G}(1,x) &= \mathrm{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{23.37}$$

Solution: We can divide $l_2(H)$ into its odd an even parts

$$D : l_2(H) \ni v \longmapsto (\{v_{2i-1}\}, \{v_{2i}\}) \in l_2(H) \oplus l_2(H) \longleftrightarrow H \oplus H. \tag{23.38}$$

and then each copy of $l_2(H)$ on the right with $H$ (using the same isometric isomorphism). Then the homotopy in the previous problem is such that

$$\tilde{G}(x, y) = D^{-1}G(y, x)D \tag{23.39}$$

accomplishes what we want.

*Problem P10.7: Eilenberg's swindle* For an infinite dimenisonal separable Hilbert space, construct an homotopy – meaning a continuous map $G : [0, 1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0, x) = L(x)$ in (23.28) and $G(1, x) = \mathrm{Id}$ and of course $G(y, 0) = G(y, 1)$ for all $x, y \in [0, 1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0, 1]$. First 'divide $H$ into 2 copies of itself' and deform from $L$ to $M(1, x)$ in (23.30). Now, 'divide the second $H$ up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi i x)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi i x)$ (with + first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

Solution: By rescaling the variables above, we now have three homotopies, always through periodic families. On $H \oplus H$ between $L(x) = e^{2\pi i x}\,\mathrm{Id}$ and the matrix

$$\begin{pmatrix} e^{4\pi i x}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{23.40}$$

Then on $H \oplus l_2(H)$ we can deform from

$$\begin{pmatrix} e^{4\pi i x}\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \text{ to } \begin{pmatrix} e^{4\pi i x}\,\mathrm{Id} & 0 \\ 0 & \tilde{G}(0, x) \end{pmatrix} \tag{23.41}$$

with $\tilde{G}(0, x)$ in (23.37). However we can then identify

$$H \oplus l_2(H) = l_2(H),\ (u, v) \longmapsto w = \{w_j\},\ w_1 = u,\ w_{j+1} = v_j,\ j \geq 1. \tag{23.42}$$

This turns the matrix of operators in (23.41) into $\tilde{G}(0, x)^{-1}$. Now, we can apply the same construction to deform this curve to the identity. Notice that this really does ultimately give an homotopy, which we can renormalize to be on $[0, 1]$ if you insist, of curves of operators on $H$ – at each stage we transfer the homotopy back to $H$.