18.102 Introduction to Functional Analysis
Spring 2009

Lecture 24. Thursday, May 7: Completeness of Hermite basis

Here is what I claim was done last time. Starting from the ground state for the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2,\ Hu_0 = u_0,\ u_0 = e^{-x^2/2} \tag{24.1}$$

and using the creation and annihilation operators

$$A = \frac{d}{dx} + x,\ C = -\frac{d}{dx} + x,\ AC - CA = 2\,\mathrm{Id},\ H = CA + \mathrm{Id} \tag{24.2}$$

I examined the higher eigenfunctions:

$$u_j = C^j u_0 = p_j(x) u_0(c),\ p(x) = 2^j x^j + \text{l.o.ts},\ Hu_j = (2j+1)u_j \tag{24.3}$$

and showed that these are orthogonal, $u_j \perp u_k$, $j \neq k$, and so when normalized give an orthonormal system in $L^2(\mathbb{R})$:

$$e_j = \frac{u_j}{2^{j/2}(j!)^{\frac{1}{2}}\pi^{\frac{1}{4}}}. \tag{24.4}$$

Now, what I want to show today, and not much more, is that the e_j form an orthonormal basis of $L^2(\mathbb{R})$, meaning they are complete as an orthonormal sequence. There are various proofs of this, but the only 'simple' ones I know involve the Fourier inversion formula and I want to use the completeness to *prove* the Fourier inversion formula, so that will not do. Instead I want to use a version of Mehler's formula. I also tried to motivate this a bit last time.

Namely, I suggested that to show the completeness of the e_j's it is enough to find a compact self-adjoint operator with these as eigenfunctions and no null space. It is the last part which is tricky. The first part is easy. Remembering that all the e_j are real, we can find an operator with the e_j;s as eigenfunctions with corresponding eigenvalues $\lambda_j > 0$ (say) by just defining

$$Au(x) = \sum_{j=0}^{\infty} \lambda_j (u, e_j) e_j(x) = \sum_{j=0}^{\infty} \lambda_j e_j(x) \int e_j(y) u(y). \tag{24.5}$$

For this to be an operator we need $\lambda_j \to 0$ as $j \to \infty$, although for convergence we just need the λ_j to be bounded. So, the problem with this is to show that A has no null space – which of course is just the completeness of the e'_j since (assuming all the λ_j are positive)

$$Au = 0 \Longleftrightarrow u \perp e_j\ \forall\ j. \tag{24.6}$$

Nevertheless, this is essentially what we will do. The idea is to write A as an *integral operator* and then work with that. I will take the $\lambda_j = w^j$ where $w \in [0, 1)$. The point is that we can find an explicit formula for

$$A_w u = \sum_{j=0}^{\infty} w^j e_j(x) e_j(y) = A(w, x, y). \tag{24.7}$$

I struggled a bit with this in class but did pretty much manage to do it.

To find $A(w,x,y)$ we use some other things I did last time. First, I defined the Fourier transform and showed its basic propertyL

$$\text{(24.8)}\quad \mathcal{F}: L^1(\mathbb{R}) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}),\ \mathcal{F}(u)=\hat{u},\\ \hat{u}(\xi)=\int e^{-ix\xi}u(x),\ \sup|\hat{u}|\le \|u\|_{L^1}.$$

Then I computed the Fourier transform of u_0, namely

$$\text{(24.9)}\quad (\mathcal{F}u_0)(\xi)=\sqrt{2\pi}u_0(\xi).$$

Now, we can use this formula, of if you like the argument to prove it, to show that

$$\text{(24.10)}\quad v=e^{-x^2/4}\Longrightarrow \hat{v}=\sqrt{\pi}e^{-\xi^2}.$$

Changing the names of the variables this just says

$$\text{(24.11)}\quad e^{-x^2}=\frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}} e^{ixs-s^2/4}ds.$$

Now, again as I discussed last time, the definition of the u_j's can be rewritten

$$\text{(24.12)}\quad u_j(x)=(-\frac{d}{dx}+x)^j e^{-x^2/2}=e^{x^2/2}(-\frac{d}{dx})^j e^{-x^2}.$$

Plugging this into (24.11) and carrying out the derivatives – which is legitimate since the integral is so strongly convergent – gives

$$\text{(24.13)}\quad u_j(x)=\frac{e^{x^2/2}}{2\sqrt{\pi}}\int_{\mathbb{R}}(-is)^j e^{ixs-s^2/4}ds.$$

Now we can use this formula twice on the sum on the left in (24.7) and insert the normalizations in (24.4) to find that

$$\text{(24.14)}\quad \sum_{j=0}^{\infty} w^j e_j(x)e_j(y)=\sum_{j=0}^{\infty}\frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}}\int_{\mathbb{R}^2}\frac{(-1)^j w^j s^j t^j}{2^j j!}e^{isx+ity-s^2/4-t^2/4}dsdt.$$

The crucial thing here is that we can sum the series to get an exponential, this allows us to finally conclude:

Lemma 19. *The identity* (24.7) *holds with*

$$\text{(24.15)}\quad A(w,x,y)=\frac{1}{\sqrt{\pi}\sqrt{1-w^2}}\exp\left(-\frac{1-w}{4(1+w)}(x+y)^2-\frac{1+w}{4(1-w)}(x-y)^2\right)$$

Proof. Summing the series in (24.14) we find that

$$\text{(24.16)}\quad A(w,x,y)=\frac{e^{x^2/2+y^2/2}}{4\pi^{3/2}}\int_{\mathbb{R}^2}\exp(-\frac{1}{2}wst+isx+ity-s^2/4-t^2/4)dsdt.$$

Now, we can use the same formula as before for the Fourier transform of u_0 to evaluate these integrals explicitly. I think the clever way, better than what I did in lecture, is to change variables by setting

$$\text{(24.17)}\quad s=(S+T)/\sqrt{2},\ t=(S-T)/\sqrt{2}\Longrightarrow dsdt=dSdT,\\ -\frac{1}{2}wst+isx+ity-s^2/4-t^2/4=iS\frac{x+y}{\sqrt{2}}-\frac{1}{4}(1+w)S^2 iT\frac{x-y}{\sqrt{2}}-\frac{1}{4}(1-w)T^2.$$

The formula for the Fourier transform of $\exp(-x^2)$ can be used, after a change of variable, to conclude that

$$\begin{aligned}\int_{\mathbb{R}} \exp(iS\frac{x+y}{\sqrt{2}} - \frac{1}{4}(1+w)S^2)dS &= \frac{2\sqrt{\pi}}{\sqrt{(1+w)}}\exp(-\frac{(x+y)^2}{2(1+w)})\\ \int_{\mathbb{R}} \exp(iT\frac{x-y}{\sqrt{2}} - \frac{1}{4}(1-w)T^2)dT &= \frac{2\sqrt{\pi}}{\sqrt{(1-w)}}\exp(-\frac{(x-y)^2}{2(1-w)}).\end{aligned} \tag{24.18}$$

Inserting these formulæ back into (24.16) gives

$$A(w,x,y) = \frac{1}{\sqrt{\pi}\sqrt{1-w^2}}\exp\left(-\frac{(x+y)^2}{2(1+w)} - \frac{(x-y)^2}{2(1-w)} + \frac{x^2}{2} + \frac{y^2}{2}\right) \tag{24.19}$$

which after a little adjustment gives (24.15). □

Now, this explicit representation of A_w as an integral operator allows us to show

Proposition 31. *For all real-valued $f \in L^2(\mathbb{R})$,*

$$\sum_{j=1}^{\infty} |(u, e_j)|^2 = \|f\|^2_{L^2}. \tag{24.20}$$

Proof. By definition of A_w

$$\sum_{j=1}^{\infty} |(u, e_j)|^2 = \lim_{w\uparrow 1}(f, A_w f) \tag{24.21}$$

so (24.20) reduces to

$$\lim_{w\uparrow 1}(f, A_w f) = \|f\|^2_{L^2}. \tag{24.22}$$

To prove (24.22) we will make our work on the integral operators rather simpler by assuming first that $f \in \mathcal{C}^0(\mathbb{R})$ is continuous and vanishes outside some bounded interval, $f(x) = 0$ in $|x| > R$. Then we can write out the L^2 inner product as a doulbe integral, which is a genuine (iterated) Riemann integral:

$$(f, A_w f) = \int\int A(w,x,y)f(x)f(y)dydx. \tag{24.23}$$

Here I have used the fact that f and A are real-valued.

Look at the formula for A in (24.15). The first thing to notice is the factor $(1-w^2)-\frac{1}{2}$ which blows up as $w \to 1$. On the other hand, the argument of the exponential has two terms, the first tends to 0 as $w \to 1$ and the second blows up, at least when $x - y \neq 0$. Given the signs, we see that

$$\begin{gathered}\text{if } \epsilon > 0,\ X = \{(x,y); |x| \le R, |y| \le R, |x-y| \ge \epsilon\} \text{ then}\\ \sup_X |A(w,x,y)| \to 0 \text{ as } w \to 1.\end{gathered} \tag{24.24}$$

So, the part of the integral in (24.23) over $|x-y| \ge \epsilon$ tends to zero as $w \to 1$.

So, look at the other part, where $|x-y| \le \epsilon$. By the (uniform) continuity of f, given $\delta > 0$ there exits $\epsilon > 0$ such that

$$|x-y| \le \epsilon \Longrightarrow |f(x) - f(y)| \le \delta. \tag{24.25}$$

Now we can divide (24.23) up into three pieces:-

$$(f, A_w f) = \int_{S\cap\{|x-y|\geq\epsilon\}} A(w,x,y)f(x)f(y)dydx + \int_{S\cap\{|x-y|\leq\epsilon\}} A(w,x,y)(f(x)-f(y))f(y)dydx + \int_{S\cap\{|x-y|\leq\epsilon\}} A(w,x,y)f(y)^2dydx \tag{24.26}$$

where $S = [-R, R]^2$.

Look now at the third integral in (24.26) since it is the important one. We can change variable of integration from x to $t = \sqrt{\frac{1+w}{1-w}}(x-y)$ and then this becomes

$$\int_{S\cap\{|x-y|\leq\epsilon\}} A(w, y + t\sqrt{\frac{1-w}{1+w}}, y)f(y)^2dydt,$$

$$A(w, y+t\sqrt{\frac{1-w}{1+w}}, y) = \frac{1}{\sqrt{\pi}(1+w)}\exp\left(-\frac{1-w}{4(1+w)}(2y+t\sqrt{1-w})^2\right)\exp\left(-\frac{t^2}{4}\right). \tag{24.27}$$

Here y is bounded; the first exponential factor tends to 1 so it is straightforward to show that for any $\epsilon > 0$ the third term in (24.26) tends to

$$\|f\|^2_{L^2} \text{ as } w \to 1 \text{ since } \int e^{-t^2/4} = 2\sqrt{\pi}. \tag{24.28}$$

Noting that $A \geq 0$ the same sort of argument shows that the second term is bounded by a constant multiple of δ. So this proves (24.22) (first choose δ then ϵ) and hence (24.20) under the assumption that f is continuous and vanishes outside some interval $[-R, R]$.

However, the general case follows by continuity since such continuous functions vanishing outside compact sets are dense in $L^2(\mathbb{R})$ and both sides of (24.20) are continuous in $f \in L^2(\mathbb{R})$. □

Now, (24.22) certainly implies that the e_j form an orthonormal basis, which is what we wanted to show – but hard work! I did it really to remind you of how we did the Fourier series computation of the same sort and to suggest that you might like to compare the two arguments.