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18.102 Introduction to Functional Analysis  
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## Lecture 25. TUESDAY, MAY 12: FOURIER TRANSFORM

Last time I showed the completeness of the orthonormal sequence formed by the eigenfunctions of the harmonic oscillator. This allows us to prove some basic facts about the Fourier transform, which we already know is a linear operator

$$(25.1) \quad L^1(\mathbb{R}) \longrightarrow C_\infty^0(\mathbb{R}), \quad \hat{u}(\xi) = \int e^{ix\xi} u(x) dx.$$

Namely we have already shown the effect of the Fourier transform on the ‘ground state’:

$$(25.2) \quad \mathcal{F}(u_0)(\xi) = \sqrt{2\pi} e_0(\xi).$$

By a similar argument we can check that

$$(25.3) \quad \mathcal{F}(u_j)(\xi) = \sqrt{2\pi} i^j u_j(\xi) \quad \forall j \in \mathbb{N}.$$

As usual we can proceed by induction using the fact that  $u_j = C u_{j-1}$ . The integrals involved here are very rapidly convergent at infinity, so there is no problem with the integration by parts in

$$(25.4) \quad \begin{aligned} \mathcal{F}\left(\frac{d}{dx} u_{j-1}\right) &= \lim_{T \rightarrow \infty} \int_{-T}^T e^{-ix\xi} \frac{du_{j-1}}{dx} dx \\ &= \lim_{T \rightarrow \infty} \left( \int_{-T}^T (i\xi) e^{-ix\xi} u_{j-1} dx + [e^{-ix\xi} u_{j-1}(x)]_{-T}^T \right) = (i\xi) \mathcal{F}(u_{j-1}), \\ \mathcal{F}(x u_{j-1}) &= i \int \frac{de^{-ix\xi}}{d\xi} u_{j-1} dx = i \frac{d}{d\xi} \mathcal{F}(u_{j-1}). \end{aligned}$$

Taken together these identities imply the validity of the inductive step:

$$(25.5) \quad \mathcal{F}(u_j) = \mathcal{F}\left(-\frac{d}{dx} + x\right) u_{j-1} = \left(i\left(-\frac{d}{d\xi} + \xi\right)\right) \mathcal{F}(u_{j-1}) = iC(\sqrt{2\pi} i^{j-1} u_{j-1})$$

so proving (25.3).

So, we have found an orthonormal basis for  $L^2(\mathbb{R})$  with elements which are all in  $L^1(\mathbb{R})$  and which are also eigenfunctions for  $\mathcal{F}$ .

**Theorem 17.** *The Fourier transform maps  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  into  $L^2(\mathbb{R})$  and extends by continuity to an isomorphism of  $L^2(\mathbb{R})$  such that  $\frac{1}{\sqrt{2\pi}} \mathcal{F}$  is unitary with the inverse of  $\mathcal{F}$  the continuous extension from  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  of*

$$(25.6) \quad \mathcal{F}(f)(x) = \frac{1}{2\pi} \int e^{ix\xi} f(\xi) d\xi.$$

*Proof.* This really is what we have already proved. The elements of the Hermite basis  $e_j$  are all in both  $L^1(\mathbb{R})$  and  $L^2(\mathbb{R})$  so if  $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  its image under  $\mathcal{F}$  because we can compute the  $L^2$  inner products and see that

$$(25.7) \quad (\mathcal{F}(u), e_j) = \int_{\mathbb{R}^2} e_j(\xi) e^{ix\xi} u(x) dx d\xi = \int \mathcal{F}(e_j)(x) u(x) dx = \sqrt{2\pi} i^j (u, e_j).$$

Now Bessel’s inequality shows that  $\mathcal{F}(u) \in L^2(\mathbb{R})$  (it is of course locally integrable) since it is continuous).

Everything else now follows easily. □

Notice in particular that we have also proved Parseval's and Plancherel's identities:-

$$(25.8) \quad \|\mathcal{F}(u)\|_{L^2} = \sqrt{2\pi}\|u\|_{L^2}, \quad (\mathcal{F}(u), \mathcal{F}(v)) = 2\pi(u, v), \quad \forall u, v \in L^2(\mathbb{R}).$$

Since we are at the end of the course, more or less, I do not have any time to remind you of the many applications of the Fourier transform. However, let me just indicate the definitions of Sobolev spaces and Schwartz space and how they are related to the Fourier transform.

First Sobolev spaces. We now see that  $\mathcal{F}$  maps  $L^2(\mathbb{R})$  isomorphically onto  $L^2(\mathbb{R})$  and we can see from (25.4) for instance that it 'turns differentiations by  $x$  into multiplication by  $\xi$ '. Of course we do not know how to differentiate  $L^2$  functions so we have some problems making sense of this. One way, the usual mathematicians' trick, is to turn what we want into a definition.

*Definition 10.* The Sobolev spaces of order  $s$ , for any  $s \in (0, \infty)$ , are defined as subspaces of  $L^2(\mathbb{R})$  :

$$(25.9) \quad H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R})\}.$$

It is natural to identify  $H^0(\mathbb{R}) = L^2(\mathbb{R})$ .

*Exercise 1.* Show that the Sobolev spaces of positive order are Hilbert spaces with the norm

$$(25.10) \quad \|u\|_s = \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\|_{L^2}.$$

Having defined these spaces, which get smaller as  $s$  increases it can be shown for instances that if  $n \geq s$  is an integer then the set of  $n$  times continuously differentiable functions on  $\mathbb{R}$  which vanish outside a compact set are dense in  $H^s$ . This allows us to justify, by continuity, the following statement:-

**Proposition 32.** *The bounded linear map*

$$(25.11) \quad \frac{d}{dx} : H^s(\mathbb{R}) \longrightarrow H^{s-1}(\mathbb{R}), \quad s \geq 1, \quad v(x) = \frac{du}{dx} \iff \hat{v}(\xi) = i\xi \hat{u}(\xi)$$

*is consistent with differentiation on  $n$  times continuously differentiable functions of compact support, for any integer  $n \geq s$ .*

One of the more important results about Sobolev spaces – of which there are many – is the relationship between these ' $L^2$  derivatives' and 'true derivatives'.

**Theorem 18** (Sobolev embedding). *If  $n$  is an integer and  $s > n + \frac{1}{2}$  then*

$$(25.12) \quad H^s(\mathbb{R}) \subset C_{\infty}^n(\mathbb{R})$$

*consists of  $n$  times continuously differentiable functions with bounded derivatives to order  $n$ .*

This is actually not so hard to prove.

These are not the only sort of spaces with 'more regularity' one can define and use. For instance one can try to treat  $x$  and  $\xi$  more symmetrically and define smaller spaces than the  $H^s$  above by setting

$$(25.13) \quad H_{\text{iso}}^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}), (1 + |x|^2)^{\frac{s}{2}} u \in L^2(\mathbb{R})\}.$$

*Exercise 2.* Find a norm with respect to which these ('isotropic Sobolev spaces')  $H_{\text{iso}}^s(\mathbb{R})$  are Hilbert spaces.

*Exercise 3.* (Probably too hard) Show that the harmonic oscillator extends by continuity to an isomorphism

$$(25.14) \quad H : H_{\text{iso}}^{s+2}(\mathbb{R}) \longrightarrow H_{\text{iso}}^s(\mathbb{R}) \quad \forall s \geq 2.$$

Finally in this general vein, I wanted to point out that Hilbert, and even Banach, spaces are not the end of the road! One very important space in relation to a direct treatment of the Fourier transform, is the Schwartz space. The definition is reasonably simple. Namely we denote Schwartz space by  $\mathcal{S}(\mathbb{R})$  and say

$$u \in \mathcal{S}(\mathbb{R}) \iff u : \mathbb{R} \longrightarrow \mathbb{C}$$

(25.15) is continuously differentiable of all orders and for every  $n$  and ,

$$\|u\|_n = \sum_{k+p \leq n} \sup_{x \in \mathbb{R}} (1+|x|)^k \left| \frac{d^p u}{dx^p} \right| < \infty.$$

All these inequalities just mean that all the derivatives of  $u$  are ‘rapidly decreasing at  $\infty$ ’ in the sense that they stay bounded when multiplied by any polynomial.

So in fact we know already that  $\mathcal{S}(\mathbb{R})$  is not empty since the elements of the Hermite basis,  $e_j \in \mathcal{S}(\mathbb{R})$  for all  $j$ .

As you can see from the definition in (25.15) this is not likely to be a Banach space. Each of the  $\|\cdot\|_n$  is a norm. However,  $\mathcal{S}(\mathbb{R})$  is pretty clearly not going to be complete with respect to any one of these. However it is complete with respect to all, countably many, norms. What does this mean? In fact  $\mathcal{S}(\mathbb{R})$  is a *metric space* with the metric

$$(25.16) \quad d(u, v) = \sum_n 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n}$$

as you can check, with some thought perhaps required. So the claim is that  $\mathcal{S}(\mathbb{R})$  is complete as a metric space – such a thing is called a Fréchet space.

What has this got to do with the Fourier transform? The point is that (25.17)

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \text{ is an isomorphism and } \mathcal{F}\left(\frac{du}{dx}\right) = i\xi\mathcal{F}(u), \quad \mathcal{F}(xu) = -i\frac{d\mathcal{F}(u)}{d\xi}$$

where this now makes sense. The Sobolev embedding theorem implies that

$$(25.18) \quad \mathcal{S}(\mathbb{R}) = \bigcap_n H_{\text{iso}}^s(\mathbb{R}).$$

The dual space of  $\mathcal{S}(\mathbb{R})$  is the space, denoted  $\mathcal{S}'(\mathbb{R})$ , of tempered distributions on  $\mathbb{R}$ .