Chapter 1

Several Complex Variables

Lecture 1

Lectures with Victor Guillemin,Texts:
Hormander: Complex Analysis in Several Variables
Griffiths: Principles in Algebraic Geometry
Notes on Elliptic Operators
No exams, 5 or 6 HW's.
Syllabus (5 segments to course, 6-8 lectures each)

1. Complex variable theory on open subsets of $\mathbb{C}^n$. Hartog, simply pseudoconvex domains, inhomogeneous C.R.
2. Theory of complex manifolds, Kaehler manifolds
3. Basic theorems about elliptic operators, pseudo-differential operators
4. Hodge Theory on Kaehler manifolds
5. Geometry Invariant Theory.

1 Complex Variable and Holomorphic Functions

U an open set in $\mathbb{R}^n$, let $C^\infty(U)$ denote the C^∞ function on U. Another notation for continuous function: Let A be any subset of $\mathbb{R}^n$, $f \in C^\infty(A)$ if and only $f \in C^\infty(U)$ with $U \supset A$, U open. That is, f is C^∞ on A if it can be extended to an open set around it.

As usual, we will identify $\mathbb{C}$ with $\mathbb{R}^2$ by $z \mapsto (x, y)$ when $z = x + iy$. On $\mathbb{R}^2$ the standard de Rham differentials are dx, dy. On $\mathbb{C}$ we introduce the de Rham differentials

$$dz = dx + idy \qquad d\bar{z} = dx - idy$$

Let U be open in $\mathbb{C}$, $f \in C^\infty(U)$ then the differential is given as follows

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial x}\left(\frac{dz + d\bar{z}}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{dz - d\bar{z}}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z} \end{aligned}$$

If we make the following definitions, the differential has a succinct form

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

so

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

We take this to be the definition of the differential operator.

Definition. $f \in \mathcal{O}(U)$ (the holomorphic functions) iff $\partial f/\partial \bar{z} = 0$. So if $f \in \mathcal{O}(U)$ then $df = \frac{\partial f}{\partial z} dz$.

Examples

1. $z \in \mathcal{O}(U)$
2. $f, g \in C^\infty(U)$ then
$$\frac{\partial f}{\partial \bar{z}} fg = \frac{\partial f}{\partial \bar{z}} g + f \frac{\partial g}{\partial \bar{z}}$$
so if $f, g \in \mathcal{O}(U)$ then $fg \in \mathcal{O}(U)$.
3. By the above two, we can say z, $z^2, \ldots$ and any polynomial in z is in $\mathcal{O}(U)$.
4. Consider a formal power series $f(z) \sim \sum_{i=1}^\infty a_i z^i$ where $|a_i| \leq (\text{const}) R^{-i}$. Then if $D = \{|z| < R\}$ the power series converges uniformly on any compact set in D, so $f \in C(D)$. And by term-by-term differentiation we see that the differentiated power series converges, so $f \in C^\infty(D)$, and the differential w/ respect to $\bar{z}$ goes to 0, so $f \in \mathcal{O}(D)$.
5. $a \in \mathcal{C}$, $f(z) = \frac{1}{z-a} \in C^\infty(\mathcal{C} - \{a\})$.

Cauchy Integral Formula

Let U be an open bounded set in $\mathbb{C}$, ∂U is smooth, $f \in C^\infty(\overline{U})$. Let $u = f dz$ by Stokes

$$\int_{\partial U} f dz = \int_U du \qquad du = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz$$

so

$$\int_{\partial U} f dz = \int_U du = \int_U \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$$

Now, take $a \in U$ and remove $D_\epsilon = \{|z-a| < \epsilon\}$, and let the resulting region be $U_\epsilon = U - \overline{D}_\epsilon$. Replace f in the above by $\frac{f}{z-a}$. Note that $(z-a)^{-1}$ is holomorphic. We get

$$\int_{\partial U_\epsilon} \frac{f}{z-a} dz = \int_{U_\epsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a} d\bar{z} \wedge dz$$

Note: The boundary of U is oriented counter-clockwise, and the inner boundary D_ϵ is oriented clockwise. When orientations are taken into account the above becomes

$$\int_{\partial U} \frac{f}{z-a} dz - \int_{\partial D_\epsilon} \frac{f(z)}{z-a} dz = \int_{U_\epsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a} d\bar{z} \wedge dz \tag{1.1}$$

The second integral, with the change of coordinates $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, $\frac{dz}{z-a} = i d\theta$. This gives

$$\int_{\partial D_\epsilon} \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a + e^{i\theta}) d\theta.$$

Now we look at what happens when $\epsilon \to 0$. Well, $\frac{1}{z-a} \in \mathcal{L}^1(U)$, so by Lebesgue dominated convergence if we let $U_\epsilon \to U$, and the integral remians unchanged. On the left hand side we get $-if(a)2\pi$, and altogether we have

$$2\pi i f(a) = \int_U \frac{f}{z-a} dz + \int_U \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a} dz \wedge d\bar{z}$$

In particular, if $f \in \mathcal{O}(U)$ then

$$\boxed{2\pi i f(a) = \int_{\partial U} \frac{f}{z-a} dz}$$

Applications:

$f \in C^\infty(\overline{U}) \cap \mathcal{O}(U)$, take $a \rightsquigarrow z$, $z \rightsquigarrow \eta$ then just rewriting

$$2\pi i f(z) = \int_{\partial U} \frac{f(\eta)}{\eta - z} d\eta$$

If we let $U = \{D : |z| < R\}$. Then

$$\frac{1}{\eta - z} = \frac{1}{\eta \left(1 - \frac{z}{\eta}\right)} = \frac{1}{\eta} \sum_{k=0}^{\infty} \frac{z^k}{\eta^k}$$

and since on boundary $|\eta| = R$, $|z| < R$ so the series converges uniformly on compact sets, we get

$$\int_{\partial U} \frac{f(\eta)}{\zeta - z} d\eta = \sum_{k=0}^{\infty} a_k z^k \qquad a_k = \int_{|\eta|=R} \frac{f(\eta)}{\eta^{k+1}} d\eta$$

or $a_k = \frac{1}{k!} \frac{\partial^k}{\partial z^k} f(0)$. This is the holomorphic Taylor expansion.

Now if we take $z \rightsquigarrow z - a$, $D : |z - a| < R$, $f \in \mathcal{O}(U) \cap C^\infty(\overline{U})$ then

$$f(z) = \sum a_k (z - a)^k \qquad a_k = \frac{1}{k!} \frac{\partial^k}{\partial z^k} f(a)$$

We can apply this a prove a few theorems.

Theorem. *U a connected open set in $\mathbb{C}$. $f, g \in \mathcal{O}(U)$, suppose there exists an open subset V of U on which $f = g$. We can conclude $f \equiv g$, this is unique analytic continuation.*

Proof. W set of all points $a \in U$ where

$$\frac{\partial^k f}{\partial z^k}(a) = \frac{\partial^k g}{\partial z^k} \qquad k = 0, 1, \ldots$$

holds. Then W is closed, and we see that W is also open, so $W = U$.