Lecture 2

Cauchy integral formula again. U an open bounded set in $\mathbb{C}$, ∂U smooth, $f \in C^\infty(\overline{U})$, $z \in U$

$$f(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{f(\eta)}{\eta - z}d\eta + \frac{1}{2\pi i}\int_U \frac{\partial f}{\partial \bar{\eta}}(\eta)\frac{1}{\eta - z}d\eta \wedge d\bar{\eta}$$

the second term becomes 0 when f is holomorphic, i.e. the area integral vanishes, and we get

$$f(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{f(\eta)}{\eta - z}d\eta$$

Now take $D : |z - a| < \epsilon$, $f \in \mathcal{O}(D) \cap C^\infty(\overline{D})$, then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + \epsilon e^{i\theta})d\theta$$

More applications:

Theorem (Maximum Modulus Principle). *U any open connected set in $\mathbb{C}$, $f \in \mathcal{O}(U)$ then if $|f|$ has a local maximum value at some point $a \in U$ then f has to be constant.*

First, a little lemma.

Lemma. *If $f \in \mathcal{O}(U)$ and $Ref \equiv 0$, then f is constant.*

Proof. Trivial consequence of the definition of holomorphic. □

Proof of Maximum Modulus Principle. Assume $f(a)$ is positive (we can do this by a trivial normalization operation). Let $u(z) = \text{Re } f$. Now from above

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + \epsilon e^{i\theta}) d\theta$$

The LHS is real valued and trivially

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a) d\theta$$

we subtract the above 2 and we get

$$0 = \int_0^{2\pi} f(a) - u(a + \epsilon e^{\theta}) d\theta.$$

When ϵ is sufficiently small, since a is a local maximum, the integral is greater than 0, $f(a) = u(a + \epsilon e^{i\theta})$ so Re f is constant in a neighborhood of a and we can normalize and assume Re $f = 0$ near a, so by analytic continuation f is constant on U. □

Inhomogeneous CR Equation

Consider U an open bounded subset of $\mathbb{C}$, ∂U a smooth boundary, $g \in C^\infty(\overline{U})$. The Inhomogeneous CR equation is the following PDE: find $f \in C^\infty(U)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g$$

The question is, does there exists a solution for arbitrary g?

First, consider another, simpler version of CR with $g \in C_0^\infty(\mathbb{C})$. Does there exists $f \in C^\infty(\mathbb{C})$ such that $\partial f / \partial \bar{z} = g$?

Lemma. *We claim the function f defined by the integral*

$$f(z) = \frac{1}{2\pi i} \int \frac{g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}$$

is in $C^\infty(\mathbb{C})$ and satisfies $\partial f / \partial \bar{z} = g$.

Proof. Perform the change of variables $w = z - \eta$, $dw = -d\eta$, $d\bar{w} = -d\bar{\eta}$ and $\eta = z - w$ then the integral above becomes

$$-\int \frac{g(z - w)}{w} dw \wedge d\bar{w} = f(z)$$

Now it is clear that $f \in C^\infty(\mathbb{C})$, because if we take $\partial / \partial z$, we can just keep differentiating under the integral. And now

$$\frac{\partial f}{\partial z} = -\frac{1}{2\pi i} \int \frac{\left(\frac{\partial g}{\partial \bar{z}}\right)(z - w)}{w} dw \wedge d\bar{w} = \frac{1}{2\pi i} \int \frac{\left(\frac{\partial g}{\partial \bar{\eta}}\right)(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}$$

Let $A = \text{supp } g$, so A is compact, then there exists U open and bounded such that ∂U is smooth and $A \subset U$. For $g \in C^\infty(\overline{U})$ write down using the Cauchy integral formula

$$g(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{g(\eta)}{\eta - z} d\eta + \frac{1}{2\pi i} \int_U \frac{\partial g}{\partial \bar{\eta}}(\eta) \frac{d\eta \wedge d\bar{\eta}}{\eta - z}$$

On ∂U, g is identically 0, so the first integral is 0. For the second integral we replace A by the entire complex plane, so

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g}{\partial \bar{\eta}}(\eta) \frac{d\eta \wedge d\bar{\eta}}{\eta - z}$$

which is the expression for $\frac{\partial f}{\partial \bar{z}}$ □

Now, we want to get rid of our compactly supported criterion. Let U be bounded, ∂U smooth and $g \in C^\infty(\overline{U})$, $\frac{\partial f}{\partial \bar{z}} = g$.

Make the following definition

$$f(z) := \frac{1}{2\pi i}\int_U \frac{g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}$$

Take $a \in U$, D an open disk about a, $\overline{D} \subset U$. Check that $f \in C^\infty$ on D and that $\partial f/\partial \bar{z} = g$ on D. Since a is arbitrary, if we can prove this we are done. Take $\rho \in C_0^\infty(U)$ so that $\rho \equiv 1$ on a neighborhood of $\overline{D}$, then

$$f(z) = \underbrace{\frac{1}{2\pi i}\int \frac{\rho(\eta)g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}}_{I} + \underbrace{\frac{1}{2\pi i}\int (1-\rho)\frac{g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}}_{II}$$

The first term, I, is in $C_0^\infty(\mathbb{C})$, so I is C^∞ on $\mathbb{C}$ and $\partial I/\partial \bar{z} = \rho g$ on $\mathbb{C}$ and so is equal to $g|_D$. We claim that $II|_D$ is in $\mathcal{O}(D)$. The Integrand is 0 on an open set containing D, so $\partial II/\partial \bar{z} = 0$ on D.

We conclude that $\partial f(z)/\partial \bar{z} = g(z)$ on D. (The same result could have just been obtained by taking a partition of unity)

Transition to Several Complex Variables

We are now dealing with $\mathbb{C}^n$, coordinatized by $z = (z_1, \ldots, z_n)$, and $z_k = x_k + iy_k$ and $dz_k = dx_k + idy_k$.

Given U open in $\mathbb{C}^n$, $f \in C^\infty(U)$ we define

$$\frac{\partial f}{\partial z_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right) \qquad \frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right)$$

So the de Rham differential is defined by

$$df = \sum\left(\frac{\partial f}{\partial x_i}dx_i + \frac{\partial y}{\partial y_i}dy_i\right) = \sum \frac{\partial f}{\partial z_k}dz_k + \sum \frac{\partial f}{\partial \bar{z}_k}d\bar{z}_k := \partial f + \bar{\partial} f$$

so $df = \partial f + \overline{\partial} f$.

Let $\Omega^1(U)$ be the space of C^∞ de Rham 1-forms, and $u \in \Omega^1(U)$ then

$$u = u' + u'' = \sum a_i dz_i + \sum b_i d\bar{z}_i \qquad a_i, b_i \in C^\infty(U)$$

we introduce the following notation

$$\Omega^{1,0} = \left\{\sum a_k dz_k, a_k \in C^\infty(U)\right\}$$
$$\Omega^{0,1} = \left\{\sum b_k d\bar{z}_k, b_k \in C^\infty(U)\right\}$$

and therefore there is a decomposition $\Omega^1(U) = \Omega^{1,0}(U) \oplus \Omega^{0,1}(U)$. We can rephrase a couple of the lines above in the following way: $df = \partial f + \overline{\partial} f$, $\partial f \in \Omega^{1,0}$, $\overline{\partial} f \in \Omega^{0,1}$.

Definition. $f \in \mathcal{O}(U)$ if $\overline{\partial} f = 0$, i.e. if $\partial f/\partial \bar{z}_k = 0$, $\forall k$.

Lemma. *For $f, g \in C^\infty(U)$, $\overline{\partial} fg = f\overline{\partial} g + g\overline{\partial} f$, thus $fg \in \mathcal{O}(U)$.*

Obviously, $z_1, \ldots, z_n \in \mathcal{O}(U)$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, then $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and $z^\alpha \in \mathcal{O}(\mathbb{C})$. Then

$$p(z) = \sum_{|\alpha| \le N} a_\alpha z^\alpha \in \mathcal{O}(\mathbb{C}^n)$$

Even more generally, suppose we have the formal power series

$$f(z) = \sum_\alpha a_\alpha z^\alpha$$

and $|a_\alpha| \le C R_1^{-\alpha_1} \ldots R_n^{-\alpha_n}$. Then let $D_k : |z_k| < R_k$ and $D = D_1 \times \cdots \times D_n$ then $f(z)$ converges on D and uniformly on compact sets in D, and by differentiation we see that $f \in \mathcal{O}(D)$.

Definition. Let $D_i : |z - a_i| < R_n$, then open set $D_1 \times \cdots \times D_n$ is called a **polydisk**.