Lecture 3

Generalizations of the Cauchy Integral Formula

There are many, many ways to generalize this, but we will start with the most obvious

Theorem. *Let $D \subseteq \mathbb{C}^n$ be the polydisk $D = D_1 \times \cdots \times D_n$ where $D_i : |z_i| < R_i$ and let $f \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ then for any point $a = (a_1, \dots, a_n)$*

$$f(a) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(z_1, \dots, z_n)}{(z_1 - a_1) \dots (z_n - a_n)} dz_1 \wedge \cdots \wedge dz_n$$

Proof. We will prove by induction, but only for the case $n = 2$, the rest follow easily. We do the Cauchy Integral formula in each variable separately

$$f(z_1, a_2) = \frac{1}{2\pi i} \int_{\partial D_2} \frac{f(z_1, z_2)}{z_2 - z_2} dz_1 \qquad f(a_1, z_n) = \frac{1}{2\pi i} \int_{\partial D_2} \frac{f(z_1, z_2)}{(z_1 - a_1)} dz_2$$

Then just plug the first into the second. □

Applications: First make the following changes $a_i \rightsquigarrow z_i$, $z_i \rightsquigarrow \eta_i$, then

$$f(z_1, \dots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\eta)}{(\eta_1 - z_1) \dots (\eta_n - z_n)} d\eta_1 \wedge \cdots \wedge d\eta_n$$

As before in the single variable case we make the following replacements

$$\frac{1}{\prod(\eta_i - z_i)} = \frac{1}{\eta_1 \dots \eta_n} \prod \frac{1}{1 - \frac{z_i}{\eta_i}} = \frac{1}{\eta_1 \dots \eta_n} \sum_\alpha \frac{z^\alpha}{\eta^\alpha}$$

for $\eta \in \partial D_1 \times \cdots \times \partial D_n$ we have uniform converge for z on compact subsets of D. So by the Lebesgue dominated convergence theorem

$$f(z) = \sum_\alpha a_\alpha z^\alpha \qquad a_\alpha = \left(\frac{1}{2\pi i}\right)^n \int \frac{f(\eta)}{\eta_1^{\alpha_1+1} \dots \eta_n^{\alpha_n+1}} d\eta_1 \wedge \cdots \wedge d\eta_n$$

Theorem. *U open in $\mathbb{C}^n$, $f \in \mathcal{O}(U)$, $a \in U$ and D a polydisk centered at a with $\overline{D} \subseteq U$ then on D we have*

$$f(z) = \sum_\alpha a_\alpha (z_1 - a_1)^{\alpha_1} \dots (z_n - a_n)^{\alpha_n}$$

(we will call this $(*)$ from now on)

Proof. Apply the previous little theorem to $f(z - a)$. □

Note we can check by differentiation that the coefficients are $a_\alpha = \frac{1}{\alpha!} \partial f / \partial z^\alpha (a)$.

Theorem. *U is a connected open set in $\mathbb{C}^n$ with $f, g \in \mathcal{O}(U)$. If $f = g$ on an open subset $V \subset U$ then $f = g$ on all of U.*

Proof. As in one dimension. □

Theorem (Maximum Modulus Principle). *U is a connected open set in $\mathbb{C}^n$, $f \in \mathcal{O}(U)$. If $|f|$ achieves a local maximum at some point $a \in U$ then f is constant*

Proof. Left as exercise. □

As a reminder:

Theorem. *Let $g \in C_0^\infty(\mathbb{C})$ then if f is the function*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}$$

then $f \in C^\infty(\mathbb{C})$ and $\partial f/\partial \bar{z} = g$.

What about the n-dimensional case? That is, given $h_i \in C_0^\infty(\mathbb{C}^n)$, $i = 1, \ldots, n$ does there exist $f \in C^\infty(\mathbb{C}^n)$ such that $\frac{\partial f}{\partial \bar{z}_i} = h_i$, $i = 1, \ldots, n$?

There clearly can't always be a solution because we have the integrability conditions

$$\frac{\partial h_i}{\partial \bar{z}_j} = \frac{\partial h_j}{\partial \bar{z}_i}$$

Theorem (Multidimensional Inhomogeneous CR equation). *If the h_i's satisfy these integrability conditions then there exists an $f \in C^\infty(\mathbb{C}^n)$ with $\partial f/\partial \bar{z}_i = h_i$. And in fact such a solution is given by*

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{h_1(\eta_1, z_2, \ldots, z_n)}{(\eta_1 - z_1)} d\eta_1 \wedge d\bar{\eta}_1$$

Proof. This just says for get about everything except the first variable.

Clearly $f \in C^\infty(\mathbb{C}^n)$ and $\partial f/\partial \bar{z}_1 = h_1$. Now $\partial f/\partial \bar{z}_i$ we compute under the integral sign and we get

$$\frac{\partial}{\partial \bar{z}_i} h_1(\eta_1, z_2, \ldots, z_n) \frac{1}{\eta_i - z_i} \in L'(\eta_1)$$

(so it is legitimate to differentiate under the integral sign). Now

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}_i} &= \frac{1}{2\pi i}\int \frac{\partial h_1}{\partial \bar{z}_j}(\eta_1, z_2, \ldots, z_n)\frac{d\eta_1 \wedge d\bar{\eta}_1}{\eta_1 - z_1} \\ &= \frac{1}{2\pi i}\int \frac{\partial h_j}{\partial \eta_1}(\eta_1, z_2, \ldots, z_n)\frac{d\eta_1 \wedge d\bar{\eta}_1}{\eta_1 - z_1} \\ &= h_j(z_1, \ldots, z_n)\end{aligned}$$

The second set is by integrability conditions, and the lat is by the previous lemma. QED. □

Let $K \Subset \mathbb{C}^n$ be a compact st. Suppose $\mathbb{C}^n - K$ is connected. Suppose $h_i \in C_0^\infty(\mathbb{C}^n)$ are supported in K.

Theorem. *If f is the function $(*)$ then $\operatorname{supp} f \subseteq K$ (unique to higher dimension). So not only do we have a solution to the ICR eqn, it is compactly supported.*

Proof. By $(*)$ $f(z_1, \ldots, z_n)$ is identically 0 when $(z_i) \gg 0$, $i > 1$, because h_i is compactly supported. Also, since $\operatorname{supp} h_i \subseteq K$ and $\partial f/\partial \bar{z}_i = h_i$ we have that $\partial f/\partial \bar{z}_i = 0$ on $\mathbb{C}^n - K$, so $f \in \mathcal{O}(\mathbb{C}^n - K)$. The uniqueness of analytic continuation we have $f \equiv 0$ on $\mathbb{C}^n - K$ (used that $\mathbb{C}^n - K$ is connected) □

Theorem (Hartog's Theorem). *Let $K \Subset U$, $U \subset \mathbb{C}^n$ is open and connected. Suppose that $U - K$ is connected. Let $f \in \mathcal{O}(U - K)$ then f extends holomorphically to all of U. THIS IS A PROPERTY SPECIFIC TO HIGHER DIMENSIONAL SPACES.*

Proof. Let $K_1 \Subset U$ so that $K \subset \operatorname{Int} K_1$, $U - K_1$ is connected. Choose $\varphi \in C^\infty(\mathbb{C}^n)$ such that $\varphi \equiv 1$ on K and $\operatorname{supp} \varphi \subset \operatorname{Int} K_1$. Let

$$v = \begin{cases}(1-\varphi)f & \text{on } U - K \\ 0 & \text{on } K\end{cases}$$

then $v \in C^\infty(U)$. And $v \equiv f$ on $U - K$. $h_i = \frac{\partial}{\partial \bar{z}_i} v$, $i = 1, \ldots, n$. One $U - K_1$, $v = f \in \mathcal{O}(U - K_1)$ so $h_i = \frac{\partial}{\partial \bar{z}_i} f$ on $U - K1$ and f is holomorphic, so this is 0, thus $h_i \in C_0^\infty(\mathbb{C}^n)$, $\operatorname{supp} h_i \subseteq K_1$ and $\frac{\partial h_i}{\partial \bar{z}_j} = \frac{\partial h_j}{\partial \bar{z}_j}$, so $\exists w \in C_0^\infty(\mathbb{C}^n)$ such that $\frac{\partial w}{\partial \bar{z}_i} = h_i$ and $\operatorname{supp} w \subseteq K_1$. Take $g = v - w$ so $w \equiv 0$ on $\mathbb{C}^n - K$, $v = f$ on $\mathbb{C}^n - K_1$, so $g = f$ on $\mathbb{C}^n - K$ and by construction

$$\frac{\partial g}{\partial \bar{z}_i} = \frac{\partial v}{\partial \bar{z}_i} - \frac{\partial w}{\partial \bar{z}_i} = h_i - \frac{\partial}{\partial \bar{z}_i} w = 0$$

so $g \in \mathcal{O}(U)$ and $g = f$ on $U - K_1$, $f \in C^\infty(U - K)$, since $U - K$ connected, by uniqueness of analytic continuation $g = f$ on $U - K$, so g is holomorphic continuation of f onto all of U. □