Lecture 5

Notes about Exercise 1

Lemma. *Let U and V be as in Theorem 1 above. $\beta \in \Omega^{0,q}(U)$, $\overline{\partial}\beta = 0$ then there exists $\alpha \in \Omega^{0,q-1}(U)$ such that $\overline{\partial}\alpha = \beta$ on V.*

Proof. Choose a polydisk W so that $\overline{V} \subset W$, $\overline{W} \subset U$. Choose $\rho \in C_0^\infty(W)$ with $\rho \equiv 1$ on a neighborhood of V. By theorem 1 there exists $\alpha_0 \in \Omega^{0,q-1}(W)$ so that $\overline{\partial}\alpha_0 = \beta$ on W. If we take

$$\alpha = \begin{cases} \rho\alpha_0 & \text{on } W \\ 0 & \text{on } U - W \end{cases}$$

then we have a solution.

We claim that the Dolbealt complex is exact on all degrees $q \geq 2$.

Lemma. *Let $V_0, V_1, V_2, \dots$ be a sequence of polydisks so that $\overline{V}_r \subset V_{r+1}$ and $\bigcup V_1 = U$. (exhaustion on U by compact polydisk). There exists $\alpha_i \in \Omega^{0,q+1}(U)$ such that $\overline{\partial}\alpha_r = \beta$ on V_r and such that $\alpha_{r+1} = \alpha_r$ on V_{r-1}.*

Proof. By the previous lemma there exists $\alpha_r \in \Omega^{0,q-1}(U)$ with $\overline{\partial}\alpha_r = \beta$ on V_r. And for α_{r+1}, α_r on V_r, $\overline{\partial}\alpha_{r+1} = \overline{\partial}\alpha_r = \beta$ on V_r, so $\overline{\partial}(\alpha_{r+1} - \alpha_r) = 0$ on V_r. Now $q \geq 2$ so we can find $\gamma \in \Omega^{0,q-1}(U)$ such that $\overline{\partial}\gamma = \alpha_{r+1} - \alpha_r$ on V_{r-1}. Then set $\alpha_{r+1}^{\text{new}} := \alpha_{r+1}^{\text{old}} + \overline{\partial}\gamma$. So $\overline{\partial}\alpha_{r+1}^{\text{new}} = \beta$ on V_{r+1}, $\alpha_{r+1}^{\text{new}} = \alpha_r$ on V_{r-1}. ∎

We get a global solution when we set $\alpha = \alpha_r$ on V_{r-1} for all r.
(**EXERCISE** Prove exactness at $q = 1$, i.e. make this argument work for $q = 1$.)
What does exactness mean for degree 1? Well

$$\beta \in \Omega^{0,1}(U) \qquad \beta = \sum f_i d\bar{z}_i \qquad f_i \in C^\infty(U)$$

We need to show that there exists $g \in \Omega^{0,0}(U) = C^\infty(U)$ so that $\overline{\partial} g = \beta$, i.e.

$$\frac{\partial g}{\partial \bar{z}_i} = f_i \qquad i = 1, \dots, n$$

So the condition that $\overline{\partial}\beta = 0$ is just the integrability conditions.

So we have to show the following. That there exists a sequence of functions $g_r \in C^\infty(U)$. $V_0 \subset V_1 \subset \cdots \subset U$ such that $\frac{\partial g_r}{\partial \bar{z}_i} = f_i$, $i = 1, \dots, n$ on V_r (easy consequence of lemma)

We can no longer say $g_{r+1} - g_r$ on V_{r-1}. But we can pick g_r such that $|g_{r+1} - g_r| < \frac{1}{2^r}$ on V_{r-1}.

Hint Choose $g_r \in C^\infty(U)$ such that $\frac{\partial g_r}{\partial \bar{z}_i} = f_i$ on V_r. Look at $g_{r+1} - g_r$ on V_r. Note that $\frac{\partial}{\partial \bar{z}_i}(g_{r+1} - g_r) = 0$ on V_r, so $g_{r+1} - g_r \in \mathcal{O}(V_r)$. On V_{r-1} we can expand by power series to get $g_{r+1} - g_r = \sum_\alpha a_\alpha z^\alpha$, and this series is actually uniformly convergent on V_{r-1}. We try to modify g_{r+1}^{old} by setting $g_{r+1}^{\text{new}} + P_N(z)$, where $P_N(z) = \sum_{|\alpha| \leq N} a_\alpha z^\alpha$

(The exercise is due Feb 25th)

More on Dolbealt Complex

For polydisks the Dolbealt complex is acyclic (exact). But what about other kinds of open sets? The solution was obtained by Kohn in 1963.

Let U be open in $\mathbb{C}$, $\varphi : U \to \mathbb{R}$ be such that $\varphi \in C^\infty(U)$.

Definition. φ is **strictly pluri-subharmonic** if for all $p \in U$ the hermitian form

$$a \in \mathbb{C}^n \mapsto \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p) a_i \overline{a}_j$$

is positive definite.

(This definition will be important later for Kaehler manifolds)

Definition. A C^∞ function $\varphi : U \to \mathbb{R}$ is an exhaustion function if it is bounded from below and if for all $c \in \mathbb{C}$

$$K_c = \{p \in U | \varphi(p) \leq c\}$$

is compact.

Definition. U is **pseudoconvex** if it possesses a strictly pluri-subharmonic exhaustion function.

Examples

1. $U = \mathbb{C}$. If we take $\varphi = |z|^2 = z\bar{z}$, $\frac{\partial \varphi}{\partial z \partial \bar{z}} = 1$.
2. $U = D \subset \mathbb{C}$
$$\varphi = \frac{1}{1 - |z|^2} \qquad \frac{\partial \varphi}{\partial z \partial \bar{z}} = \frac{1 + |z|^2}{(1 - |z|^2)^3} > 0$$
3. $U \subset \mathbb{C}$, $U = D - \{0\} = D^o$, i.e. the punctured disk
$$\varphi^o = \frac{1}{1 - |z|^2} + \operatorname{Log} \frac{1}{|z|^2} \qquad \frac{\partial \varphi^o}{\partial z \partial \bar{z}} = \frac{\partial \varphi}{\partial z \partial \bar{z}}$$
because Log is harmonic. Note the extra term in φ^o is so the function will blow up at its point of discontinuity.

4. $\mathbb{C}^n \supset U = D_1 \times \cdots \times D_n$, where $D_i = |z_i|^2 < 1$. Take

$$\varphi = \sum \frac{1}{1 - |z_i|^2}$$

5. $\mathbb{C}^n \supset U$, $D_1^o \times \cdots \times D_k^o \times D_{k+1} \times \cdots \times D_n$

$$\varphi^o = \varphi + \sum_{i=1}^{k} \operatorname{Log} \frac{1}{|z_i|^2}$$

6. $U \subseteq \mathbb{C}^n$, $U = B^n$, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$.

$$\varphi = \frac{1}{1 - |z|^2} \qquad \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = \frac{\delta_{ij}}{(1 + |z|^2)} + \frac{2 z_i \bar{z}_j}{(1 - |z|^2)^3}$$

Theorem. *If $U_i \subset \mathbb{C}^n$, $i = 1, 2$ is pseudo-convex then $U_1 \cap U_2$ is pseudo-convex*

Proof. Take φ_i to be strictly pluri-subharmonic exhaustion functions for U_i. Then set $\varphi = \varphi_1 + \varphi_2$ on $U_1 \cap U_w$. □

Punchline:

Theorem. *The Dolbealt complex is exact on U if and only if U is pseudo-convex.*

This takes 150 pages to prove, so we'll just take it as fact.
The Dolbealt complex is the left side of the bi-graded de Rham complex.
There is another interesting complex. For example if we let $A^0 = \ker \overline{\partial} : \Omega^{p,0} \to \Omega^{p,1}$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$ and $\omega \in A^r$ then $\partial \omega \in A^{r+1}$ and we get a complex

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \cdots$$