Chapter 2

Complex Manifolds

Lecture 7

Complex manifolds

First, lets prove a holomorphic version of the inverse and implicit function theorem.

For real space the inverse function theorem is as follows: Let U be open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a C^∞ map. For $p \in U$ and for $x \in B_\epsilon(p)$ we have that

$$f(x) = \underbrace{f(p) + \frac{\partial f}{\partial x}(p)(x-p)}_{I} + \underbrace{O(|x-p|^2)}_{II}$$

I is the linear approximation to f at p.

Theorem (Real Inverse Function Theorem). *If I is a bijective map $\mathbb{R}^n \to \mathbb{R}^n$ then f maps a neighborhood U_1 of p in U diffeomorphically onto a neighborhood V of $f(p)$ in $\mathbb{R}^n$.*

Now suppose U is open in $\mathbb{C}^n$, and $f : U \to \mathbb{C}^n$ is holomorphic, i.e. if $f = (f_1, \ldots, f_n)$ then each of the f_i are holomorphic. For z close to p use the Taylor series to write

$$f(z) = \underbrace{f(p) + \frac{\partial f}{\partial z}(p)(z-p)}_{I} + \underbrace{O(|z-p|^2)}_{II}$$

I is the linear approximation of f at p.

Theorem (Holomorphic Inverse Function Theorem). *If I is a bijective map $\mathbb{C}^n \to \mathbb{C}^n$ then f maps a neighborhood U_1 of p in U biholomorphically onto a neighborhood V of $f(p)$ in $\mathbb{C}^n$.*

(biholomorphic: inverse mapping exists and is holomorphic)

Proof. By usual inverse function theorem f maps a neighborhood U_1 of p is U diffeomorphically onto a neighbrohood V of $f(p)$ in $\mathbb{C}^n$, i.e. $g = f^{-1}$ exists and is C^∞ on V. Then $f^* : \Omega^1(V) \to \Omega^1(U_1)$ is bijective and f is holomorphic, so $f^* : \Omega^1(V) \to \Omega^1(U_1)$ preserves the splitting $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$. However, if $g = f^{-1}$ then $g^* : \Omega^1(U_1) \to \Omega^1(V)$ is just $(f^*)^{-1}$ so it preserves the splitting. By a theorem we proved last lecture g has to be holomorphic. ☐

Now, the implicit function theorem.

Let U be open in $\mathbb{C}^n$ and $f_1, \ldots, f_k \in \mathcal{O}(U)$, $p \in U$.

Theorem. *If $df_1, \ldots, df_k$ are linearly independent at p, there exists a neighborhood U_1 of p in U and a neighborhood V of 0 in $\mathbb{C}^n$ and a biholomorphism $\varphi : (V, 0) \to (U_1, p)$ so that*

$$\varphi^* f_i = z_i \qquad i = 1, \ldots, k$$

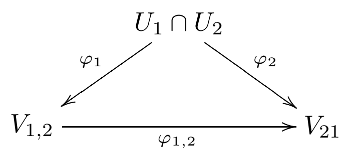
Proof. We can assume $p = 0$ and assume $f_i = z_i + O(|z|^2)$ $i = 1, \ldots, k$ near 0. Take $\psi : (U, 0) \to (\mathbb{C}^n, 0)$ given by $\psi(f_1, \ldots, f_k z_{k+1}, \ldots, z_n)$. By definition $\partial\psi/\partial z(0) = Id = [\delta_{ij}]$. ψ maps a neighborhood U_1 of 0 in U biholomorphically onto a neighborhood V of 0 in $\mathbb{C}^n$ and for $1 \leq i \leq k$, $\psi^* z_i = f_i$. Define $\varphi = \psi^{-1}$, then $\varphi^* f_i = z_i$. ∎

Manifolds

X a Hausdorff topological space and 2nd countable (there is a countable collection of open sets that defines the topology).

Definition. A **chart** on X is a triple (φ, U, V), U open in X, V an open set in $\mathbb{C}^n$ and $\varphi : U \to V$ homeomorphic.

Suppose we are given a pair of charts (φ_i, U_i, V_i), $i = 1, 2$. Then we have the overlap chart



where $\varphi_1(U_1 \cap U_2) = V_{1,2}$ and $\varphi_2(U_1 \cap U_2) = V_{2,1}$.

Definition. Two charts are **compatible** if $\varphi_{1,2}$ is biholomorphic.

Definition. An **atlas** $\mathcal{A}$ on X is a collection of mutually compatible charts such that the domains of these charts cover X.

Definition. An atlas is **complete** if every chart which is compatible with the members of $\mathcal{A}$ is in $\mathcal{A}$.

The completion operation is as follows: Take $\mathcal{A}_0$ to be any atlas then we take $\mathcal{A}_0 \rightsquigarrow \mathcal{A}$ by adding all charts compatible with $\mathcal{A}_0$ to this atlas.

Definition. A complex n-dimensional manifold is a pair $(X, \mathcal{A})$, where X is a second countable Hausdorff topological space, $\mathcal{A}$ is a complete atlas.

From now on if we mention a chart, we assume it belongs to some atlas $\mathcal{A}$.

Definition. (φ, U, V) a chart, $p \in U$ and $\varphi(p) = 0 \in \mathbb{C}^n$, then "$\varphi$ is centered at p".

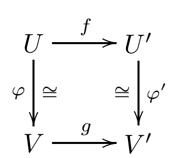
Definition. (φ, U, V) a chart and $z_1, \ldots, z_n$ the standard coordinates on $\mathbb{C}^n$. Then

$$\varphi_i = \varphi^* z_i$$

$\varphi_1, \ldots, \varphi_n$ are coordinate functions on U. We call $(U, \varphi_1, \ldots, \varphi_n)$ is a **coordinate patch**

Suppose X is an n-dimensional complex manifold, Y an m-dimensional complex manifold and $f : X \to Y$ continuous.

Definition. f is holomorphic at $p \in X$ if there exists a chart (φ, U, V) centered at p and a chart (φ', U', V') centered at $f(p)$ such that $f(U) \subset U'$ and such that in the diagram below the bottom horizontal arrow is holomorphic



(Check that this is an intrinsic definition, i.e. doesn't depend on choice of coordinates). From now on $f : X \to \mathbb{C}$ is holomorphic iff $f \in \mathcal{O}(X)$ (just by definition)

(φ, U, V) is a chart on X, V is by definition open in $\mathbb{C}^n = \mathbb{R}^{2n}$. So (φ, U, V) is a 2n-dimensional chart in the real sense. If two charts (φ_i, U_i, V_i), $i = 1, 2$ are 18.117 compatible then they are compatible in the 18.965 sense (because biholomorphisms are diffeomorphisms)

So every n-dimensional complex manifold is automatically a $2n$-dimensional C^∞ manifold. One application of this observation:

Let X be an $\mathbb{C}$-manifold, X is then a $2n$-dimensional C^∞ manifold. If $p \in X$, then T_pX the tangent space to X (as a C^∞ $2n$-dimensional manifold). T_0X is a $2n$-dimensional vector space over $\mathbb{R}$.

We claim: T_pZ has the structure of a complex n-dimensional vector space. Take a chart (φ, U, V) centered at p, so $\varphi : U \to V$ is a C^∞ diffeomorphism.

Take $(d\varphi)_p : T_p \to T_0\mathbb{C}^n = \mathbb{C}^n$. Define a complex structure on T_pX by requiring $d\varphi_p$ to be $\mathbb{C}$-linear. (check that this in independent of the choice of φ).

From the overlap diagram we get something like

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle\varphi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\varphi'} \\ V & \xrightarrow{g} & V' \end{array}$$

$$\begin{array}{ccccc} & & T'_p & & \\ & {\scriptstyle(d\varphi_1)_p}\swarrow & & \searrow{\scriptstyle(d\varphi_2)_p} & \\ T_0\mathbb{C}^n & & \xrightarrow{d\varphi_{1,2}} & & T_0\mathbb{C}^n \\ \downarrow & & & & \downarrow \\ \mathbb{C}^n & & \xrightarrow{L} & & \mathbb{C}^n \end{array} \qquad L = \left[\frac{\partial \varphi_{1,2}}{\partial z}\right]$$

X, Y, $f : X \to Y$ holomorphic, $f(p) = q$. By 18.965, $df_p : T_p \to T_q$ check that df_p is $\mathbb{C}$-linear.