Lecture 8

We'll just list a bunch of definitions. X a topological Hausdorff space, second countable.

Definition. A **chart** is a trip (φ, U, V), U open in X, V open in $\mathbb{C}$ and $\varphi : U \to V$ a homeomorphism.

If you consider two charts (φ_i, U_i, V_i), $i = 1, 2$ we get an overlap diagram. Charts are compatible if and only if the transition maps in the overlap diagram (see above) are biholomorphic.

Definition. A **atlas** is a collection $\mathcal{A}$ of charts such that

1. The domains are a cover of X
2. All members of $\mathcal{A}$ are compatible.

Definition. An atlas $\mathcal{A}$ is a **maximal atlas** then $(X, \mathcal{A})$ is a complex n-dimensional manifold.

Remark: If every open subset of X is a complex n-dimensional manifold we say $\mathcal{A}_U$ is a member of $\mathcal{A}$ with domain contained in U.
If X is a complex n-dimensional manifold it is automatically a real C^∞ $2n$-dimensional manifold.

Definition. X, Y are complex manifolds, $f : X \to Y$ is holomorphic if locally its holomorphic.

$f \in \mathcal{O}(X)$, $f : X \to \mathbb{C}$. Note if $f : X \to Y$, $g : Y \to Z$ holomorphic, then $f \circ g : X \to Z$ is as well.
Take X to be an n-dimensional complex manifolds, if we think of X as a C^∞ $2n$-dimensional then T_pX is well defined. But we showed that T_pX has a complex structure. $f : X \to Y$ holomorphic, $p \in X$, $q = f(p)$ in the real case $df_p : T_p \to T_q$, but we check that this is also $\mathbb{C}$-linear.
Notion of Charts Revisited A chart (from now on) is a triple (φ, U, V), U open in X, V open in $\mathbb{C}^n$, $\varphi : U \to V$ a biholomorphic map.

Definition. A **coordinate patch** in X is an n-tuple $(U, w_1, \ldots, w_n)$ where U is open in X and $w_i \in \mathcal{O}(U)$ such that the map $\varphi : U \to \mathbb{C}^n$

$$p \mapsto (w_1(p), \ldots, w_n(p))$$

is a biholomorphic map onto an open set V of $\mathbb{C}^n$.

Charts and coordinate patches are equivalent.

Theorem (Implicit Function Theorem in Manifold Setting). *X^n a manifold. $U_0 \subseteq X$ is an open set, $f_1, \ldots, f_k \in \mathcal{O}(U_0)$, $p \in U_0$. Assume $df_1, \ldots, df_k$ are linearly independent at p. Then there exists a coordinate patch $(U, w_1, \ldots, w_n)$, $p \in U$, $U \subset U_0$ such that $w_i = f_i$ for $i = 1, \ldots, k$.*

Proof. We can assume U_0 is the domain of the chart (U_0, V, φ), V an open set in $\mathbb{C}^n$, $\varphi : U_0 \to V$ a biholomorphism. Then just apply last lecture version of implicity function theorem to $f_i \circ (\varphi^{-1})$. □

Submanifolds

X a complex n-dimensional manfiolds. $Y \subset X$ a subset.

Definition. Y is a k-dimensional submanifold of X if for every $p \in Y$ there exists a coordinate patch $(U, z_1, \dots, z_n)$ with $p \in U$ such that $Y \cap U$ is defined by the equation $z_{k+1} = \cdots = z_n = 0$.

Remarks: A k dimensional submanifold of X is a k-dimensional complex manifold in its own right.
Call a coordinate patch with the property above an **adapted** coordinated for X. The collection of $(n+1)$-tuples $(U', z'_1, \dots, z'_k)$, $(U, z_1, \dots, z_n)$, $U' = U \cap Y$, $z'_i = z_i \mid_{U'}$ gives an atlas for X.
By the implicit function theorem this definition is equivalent to the following weaker definition.

Definition. Y is a k-dimensional submanifold X if for every $p \in Y$ there exists an open set U of p in X and $f_i \in \mathcal{O}(U)$ where $i = 1, \dots, l$, $l = n - k$ such that $df_1, \dots, df_l$ are linearly independent at p and $Y \cap U$, $f_1 = \cdots = f_l = 0$, i.e. locally Y is cut-out by l independent equation.

Examples

Affine non-singular algebraic varieties in $\mathbb{C}^n$. These are X-dimensional submanifolds, Y of $\mathbb{C}^n$ such that for every $p \in Y$ the f_i's figuring into the equation above (the ones that cut-out the manifold) are polynomials.

Projective counterparts We start by constructing the projective space $\mathbb{C}P^n$. Start with $\mathbb{C}^{n+1} - \{0\}$. Given 2 $(n+1)$-tuples we say

$$(z_0, z_1, \dots, z_n) \sim (z'_0, z'_1, \dots, z'_n)$$

in $\mathbb{C}^n - \{0\}$ if there exists $\lambda \in \mathbb{C} - \{0\}$ with $z'_i = \lambda z_i$, $i = 0, \dots, n$. $[z_0, z_1, \dots, z_n]$ are equivalence classes. We define $\mathbb{C}P^n$ to be these equivalence classes $\mathbb{C}^{n+1} - \{0\}/\sim$.

We make this into a topological space by $\pi : C^{n+1} - \{0\} \to \mathbb{C}P^n$, which is given by

$$(z_0, z_1, \dots, z_n) \sim [z_0, z_1, \dots, z_n]$$

We topologize $\mathbb{C}P^n$ by giving it the weakest topology that makes π continuous, i.e. $U \subseteq \mathbb{C}P^n$ is open if $\pi^{-1}(U)$ is open.

Lemma. *With this topology $\mathbb{C}P^n$ is compact.*

Proof. Take

$$\mathbb{S}^{2n+1} = \{(z_0, \dots, z_n) | |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

and we note

$$\pi(\mathbb{S}^{2n+1}) = \mathbb{C}P^n$$

so its the image of a compact set under a continuous map, so its compact. □

Lemma. $\mathbb{C}P^n$ *is a complex n-manifold.*

Proof. Define the standard atlas for $\mathbb{C}P^n$. For $i = 0, \dots, n$ take

$$U_i = \{[z_0, \dots, z_n] \in \mathbb{C}P^n, z_i \neq 0\}$$

Take $V_i = \mathbb{C}^n$ and define a map $\varphi_i : U_i \to V_i$ by

$$[z_0, \dots, z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i} \right)$$

$\varphi_i^{-1} : \mathbb{C}^n \to U_i$ is given by

$$(w_1, \ldots, w_n) \mapsto [w_1, \ldots, 1, \ldots, w_n]$$

where w_1 is in the 0th place, and 1 is in the ith place. The overlap diagrams for U_0 and U_1 are given by

$$\begin{array}{ccc} & U_0 \cap U_1 & \\ {\scriptstyle \varphi_0} \swarrow & & \searrow {\scriptstyle \varphi_1} \\ V_{0,1} & \xrightarrow{\varphi_{0,1}} & V_{1,0} \end{array}$$

We can check that $V_{0,1} = V_{1,0} = \{(z_1, \ldots, z_n), z_i \neq 0\}$. Also check that

$$\varphi_{0,1} : V_{0,1} \to V_{1,0} \qquad (z_1, \ldots, z_n) \mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1} \right)$$

This standard atlas gives a complex structure for $\mathbb{C}P^n$. $\square$