Lecture 10

IF $(U, z_1, \ldots, z_n)$ is a coordinate patch, then this splitting agrees with our old splitting. Son on a complex manifold we have the bicomplex $(\Omega^{*,*}, \partial, \overline{\partial})$. Again, we have lots of interesting subcomplexes.

$$A^p(X) = A^p = \ker \overline{\partial} : \Omega^{p,0} \longrightarrow \Omega^{p,1}$$

the complex of holomorphic p-forms on X, i.e. on a coordinate patch $\omega \in A^p(U)$

$$\omega = \sum f_I dz_I \qquad f_I \in \mathcal{O}(U)$$

Now, for the complex $A^p(X)$ we can compute its cohomology. There are two approaches to this

1. Hodge Theory
2. Sheaf Theory

We'll talk about sheaves fora bit.

Let X be a topological space. Top(X) is the category whose objects are open subsets of X and morphisms are the inclusion maps.

Definition. A **pre-sheaf** of abelian groups is a contravariant functor $\mathcal{F}$ from Top(X) to the category of abelian groups.

In english: $\mathcal{F}$ attached to every open set $U \subset X$ an abelian group $\mathcal{F}(U)$ and to every pair of open sets $U \supset V$ a restriction map $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$.

The functorality of this is that if $U \supset V \supset W$ then $r_{U,W} = r_{V,W} \cdot r_{U,V}$.

Examples

1. The pre-sheaf C, $U \to C(U) =$ the continuous function on U. Then the restrictions are given by
$$r_{U,V} : C(U) \to C(V) \qquad C(U) \ni f \mapsto f\mid_V \in C(V)$$
2. X a C^∞ manifold. The pre-sheaf of C^∞ functions, $U \to C^\infty(U)$. $r_{U,V}$ are as in 1.
3. Ω^r is a pre-sheaf, $U \to \Omega^r(U)$. Restriction is the usual restriction.
4. X a complex manifold, then $\Omega^{p,q}$, $U \to \Omega^{p,q}(U)$ is a pre-sheave.
5. X a complex manifold, then you have the sheaf $U \to \mathcal{O}(U)$.

Consider the pre-sheaf of C^∞-functions. Let $\{U_i\}$ be a collection of open set n X and $U = \bigcup U_i$. We claim that C^1 has the following "gluing property":

Given $f_i \in C^\infty(U_i)$ suppose

$$r_{U_i, U_i \cap U_j} f_i = r_{U_j, U_i \cap U_j} f_j$$

i.e. $f_i = f_j$ on $U_i \cap U_j$. Then there is a unique $f \in C^\infty(U)$ such that

$$r_{U,U_i} f = f_i$$

Definition. A pre-sheaf $\mathcal{F}$ is a **sheaf** if it has the gluing property.

(Note that all of all pre-sheaves in the examples are sheaves)

Sheaf Cohomology

Let $U = \{U_i, i \in I\}$, I an index set, U_i an open cover of X. Let $J = (j_0, \dots, j_k) \in I^{k+1}$, then define

$$U_J = U_{j_0} \cap \cdots \cap U_{j_k}$$

Take $N^k \subseteq I^{k+1}$ and let us say that $J \subset N^k$ if and only if $U_J \neq \emptyset$ and take

$$N = \bigsqcup N^k$$

then this is a graded set called the **nerve** of the cover U_i. N^k is called the **k-skeleton** of N.

Let $\mathcal{F}$ be the sheaf of abelian groups in X

Definition. A Cech cochain, c of degree k, with values in $\mathcal{F}$ is a map that assigns to every $J \in N^k$ an element $c(J) \in \mathcal{F}(U_J)$.

Notation. $J \in N^k$, $J = (j_0, \dots, j_k)$ and $j_i \in I$ for all $0 \leq i \leq k$. Then define

$$J_i = (j_0, \dots, \widehat{j_i}, \dots, j_k)$$

then $J_i \in N^{k-1}$ and let $r_i = r_{U_{J_i}, U_J}$.

We can define an coboundary operator

$$\delta : C^{k-1}(U, \mathcal{F}) \to C^k(U, \mathcal{F})$$

For $J \in N^k$ and $c \in C^{k-1}$ define

$$\delta c(J) = \sum_i (-1)^i r_i c(J_i)$$

(note that this makes sense, because $c(J_i) \in \mathcal{F}(U_{J_i})$.

Lemma. $\delta^2 = 0$*, i.e.* δ *is in fact a coboundary operator.*

Proof. $J \in N^{k+1}$ then

$$\begin{aligned}(\delta\delta c)(J) &= \sum_i (-1)^i r_i \delta c(J_i) \\ &= \sum_i (-1)^i r_i r_j \sum_{j<i} (-1)^j c(J_{i,j}) + \\ &\sum_i (-1)^i r_i r_j \sum_{j>i} (-1)^{j-1} c(J_{i,j})\end{aligned}$$

this is symmetric in i and j, so its 0. □

Because δ is a coboundary operator we can consider $H^k(\mathcal{U}, \mathcal{F})$, the cohomology groups of this complex.

What is $H^0(U, \mathcal{F})$? Consider $c \in C^0(U, \mathcal{F})$ then every $i \in I$, $c(i) = f_i \in \mathcal{F}(U_i)$. If $\delta c = 0$ then $r_i f_j = r_j f_i$ for all i, j. Then the gluing property of $\mathcal{F}$ tells us that there exists an $f \in \mathcal{F}(X)$ with $r_i f = f_i$, so we have proved that $H^0(X, \mathcal{F}) = \mathcal{F}(X)$, the global sections of the sheaf.

For today, we'll just compute $H^k(U, C^\infty) = 0$ for all $k \geq 1$. The proof is a bit sketchy.

Let $\{\rho_r\}_{r \in I}$ be a partition of unity subordinate to $\{U_i, i \in I\}$. Then $\rho_r \in C_0^\infty(U_r)$ and $\sum \rho_r = 1$ by definition. Given $J \in N^{k-1}$ let $(r, J) = (r, j_0, \dots, j_{k-1})$ and define a coboundary operator

$$Q : C^k(U, \mathcal{F}) \to C^{k-1}(U, \mathcal{F})$$

Take $c \in C^k$, $J \in N^{k-1}$ then

$$Qc(J) = \sum \rho_r c(r, J) \qquad \in C^\infty(U_J)$$

Explanation: First notice that (r, J) may not be in N^k. But in this case U_r and U_J are disjoint, so $\rho_r \equiv 0$ on U_J, so we just make these terms 0. What if $(r, J) \in N^k$ then $c(r, J) \in C^\infty(U_r \cap U_J)$ (but we want $Qc(J)$ to be $C^\infty(U_J)$.

But

$$\rho_r c(r, J) = \begin{cases} \rho_r c(r, J) & \text{on } U_r \cap U_J \\ 0 & \text{on } U_J - (U_r \cap U_J) \end{cases}$$

and $\rho_r \in C^\infty(U_r)$.

Proposition. *$\delta Q + Q\delta = id$.*

Corollary. *$H^k(U, C^\infty) = 0$.*

The same argument works for the sheaves Ω^*, $\Omega^{p,q}$, but NOT however for $\mathcal{O}$.