Lecture 11

U open in $\mathbb{C}^n$, $\rho \in C^\infty(U)$, $\rho : U \to \mathbb{R}$ ten ρ is strictly plurisubharmonic if for all $p \in U$ the matrix

$$\left[\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p)\right]$$

is positive definite.

If U, V open in $\mathbb{C}^n$ then $\varphi : U \to V$ is biholomorphic then for $\rho \in C^\infty(V)$ strictly plurisubharmonic $\varphi^*\rho$ is also strictly plurisubharmonic. If $q = \varphi(p)$

$$\frac{\partial^2}{\partial z_i \bar{z}_j}\varphi^*\rho(q) = \sum_{k,l} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_l}\frac{\partial \varphi_k}{\partial z_l}\frac{\partial \bar{\varphi}_l}{\partial \bar{z}_j}$$

the RHS being s.p.s.h implies the right hand side is also.

Definition. U open in $\mathbb{C}^n$ is **pseudo-convex** if it admits a s.p.s.h exhaustion function. We discussed the examples before (in particular if U_1, U_2 pseudo-convex, $U_1 \cap U_2$ is pseudo-convex)

The observation above gives that pseudoconvexity is invariant under biholomorphism.

Theorem (Hormander). *U pseudo-convex then the Dolbeault complex on U is exact.*

Back to Cech Cohomology

X a complex n-dimensional manifold and $\mathcal{U} = \{U_i, i \in I\}$ and $\mathcal{F}$ a sheaf of abelian groups. We get the Cech complex

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

and $H^p(\mathcal{U}, \mathcal{F})$ is the cohomology group of the Cech complex. We proved earlier that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$. Also, we showed that if $\mathcal{F}$ is one of the sheaves that we discussed $H^p(\mathcal{U}, \mathcal{F}) = 0, p > 0$ i.e. $\mathcal{F} = C^\infty$, Ω^r, $\Omega^{p,q}$.

But what we're really interested in is $\mathcal{F} = \mathcal{O}$.

Definition. $\mathcal{U} = \{U_i, i \in I\}$ is a pseudoconvex cover if for each i, U_i is biholomorphic to a pseudoconvex open set of $\mathbb{C}^n$.

Theorem. *If $\mathcal{U}$ is a pseudoconvex cover then the Cech cohomology groups $H^p(\mathcal{U}, \mathcal{O})$ are identified with the cohomology groups of the Dolbeault complex*

$$\Omega^{0,0}(X) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X) \xrightarrow{\bar{\partial}} \Omega^{0,2}(X) \xrightarrow{\bar{\partial}} \cdots$$

This is pretty nice, because its a comparison of very different objects. We do a proof by diagram chasing. The rows of this diagram are

$$0 \xrightarrow{\delta} \Omega^{0,q}(X) \xrightarrow{\delta} C^0(\mathcal{U}, \Omega^{0,q}) \xrightarrow{\delta} C^1(\mathcal{U}, \Omega^{0,q}) \xrightarrow{\delta} \cdots$$

To figure out the columns we have to create another way looking at the Cech complex.

Let N be the nerve of $\mathcal{U}$, $J \in N^p$, $c \in C^p(\mathcal{U}, \Omega^{0,q})$ iff c assigns to J an element $c(J) \in \Omega^{0,q}(U_J)$.

Define $\overline{\partial} c \in C^p(\mathcal{U}, \Omega^{0,q+1})$ by

$$\overline{\partial} c(J) = \overline{\partial}(c(J))$$

now $\overline{\partial} : C^p(\mathcal{U}, \Omega^{0,q}) \to C^p(\mathcal{U}, \Omega^{0,q+1})$ and we can show that $\overline{\partial}^2 = 0$.

Its not hard to show that the diagram below commutes.

$$\begin{array}{ccc}
C^p(\mathcal{U}, \Omega^{0,q}) & \xrightarrow{\delta} & C^{p+1}(\mathcal{U}, \Omega^{0,q}) \\
\overline{\partial} \downarrow & & \downarrow \overline{\partial} \\
C^p(\mathcal{U}, \Omega^{0,q+1}) & \xrightarrow{\delta} & C^{p+1}(\mathcal{U}, \Omega^{0,q+1})
\end{array}$$

Consider the map $C^p(\mathcal{U}, \Omega^{0,0}) \xrightarrow{\overline{\partial}} C^p(\mathcal{U}, \Omega^{0,1})$, what is the kernel of $\overline{\partial}$. $c \in C^p(\mathcal{U}, \Omega^{0,0})$, $J \in N^p$, $c(J) \in C^\infty(U_J)$ and $\overline{\partial} c(J) = 0$ then $c(J) \in \mathcal{O}(U_J)$. So we can extend the arrow that we are considering as follows

$$C^p(\mathcal{U}, \mathcal{O}) \xrightarrow{i} C^p(\mathcal{U}, \Omega^{0,0}) \xrightarrow{\overline{\partial}} C^p(\mathcal{U}, \Omega^{0,1}) \longrightarrow \cdots$$

Theorem. *The following sequence is exact*

$$C^p(\mathcal{U}, \Omega^{0,0}) \xrightarrow{\overline{\partial}} C^p(\mathcal{U}, \Omega^{0,1}) \xrightarrow{\overline{\partial}} \cdots$$

Observation: $J \in N^p$. The set U_J is biholomorphic to a pseudoconvex open set in $\mathbb{C}^n$. Why? U_J is non-empty and it is the intersection of pseudoconvex sets, and so it is also pseudoconvex.

Suppose we have $c \in C^p(\mathcal{U}, \Omega^{0,q})$ and $\overline{\partial} c = 0$. For $J \in N^p$, $c(J) \in C^\infty(U_J)$ and $\overline{\partial} c(J) = 0$. So there is an $f_J \in \Omega^{0,q+1}$ such that $\overline{\partial} f_I = c(J)$. Now define $c' \in C^p(\mathcal{U}, \Omega^{0,q-1})$ by $c'(J) = f_I$. Then $\overline{\partial} c' = c$.

Now, for the diagram. Set $C^{p,q} = C^p(\mathcal{U}, \Omega^{0,q})$, and $A^q = \Omega^{0,q}(X)$, $B^p = C^p(\mathcal{U}, \mathcal{O})$. We get the following diagram

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
A^3 & \xrightarrow{i} & C^{0,3} & \xrightarrow{\delta} & C^{1,3} & \xrightarrow{\delta} & C^{2,3} & \xrightarrow{\delta} & C^{3,3} & \xrightarrow{\delta} & \cdots \\
 & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
A^2 & \xrightarrow{i} & C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & C^{3,2} & \xrightarrow{\delta} & \cdots \\
 & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
A^1 & \xrightarrow{i} & C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & C^{3,1} & \xrightarrow{\delta} & \cdots \\
 & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
A^1 & \xrightarrow{i} & C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \xrightarrow{\delta} & C^{2,0} & \xrightarrow{\delta} & C^{3,0} & \xrightarrow{\delta} & \cdots \\
 & & i\uparrow & & i\uparrow & & i\uparrow & & i\uparrow & & \\
 & & B^0 & & B^1 & & B^2 & & B^3 & &
\end{array}$$

All rows except the bottom row are exact, all columns except the the left are exact. The bottom row computes $H^p(\mathcal{U}, \mathcal{O})$ and the left hand column computes $H^q(X, \text{Dolbeault})$. We need to prove that the cohomology of the bottom row is the cohomology of the left.

Hint: Take $[a] \in H^k(X, \text{Dolbeault})$, $a \in A^k = \Omega^{0,k}(X)$. The we just diagram chase down and to the right, eventually we get down to a $[b] \in H^k(\mathcal{U}, \mathcal{O})$. We have to prove that this case $[a] \rightsquigarrow [b]$ is in fact a mapping (we do this by showing that the chasing does not change cohomology class) and we have to show that the map created is bijective, which is not too hard.