Chapter 3

Symplectic and Kaehler Geometry

Lecture 12

Today: Symplectic geometry and Kaehler geometry, the linear aspects anyway.

Symplectic Geometry

Let V be an n dimensional vector space over $\mathbb{R}$, $B : V \times V \to \mathbb{R}$ a bilineare form on V.

Definition. B is alternating if $B(v, w) = -B(w, v)$. Denote by $\mathrm{Alt}^2(V)$ the space of all alternating bilinear forms on V.

Definition. Take any $B \in \mathrm{Alt}(V)$, U a subspace of V. Then we can define the orthogonal complement by

$$U^\perp = \{v \in V, B(u, v) = 0, \forall u \in U\}$$

Definition. B is non-degenerate if $V^\perp = \{0\}$.

Theorem. *If B is non-degenerate then $\dim V$ is even. Moeover, there exists a basis $e_1, \dots, e_n, f_1, \dots, f_n$ of V such that $B(e_i, e_n) = B(f_i, f_j) = 0$ and $B(e_i, f_j) = \delta_{ij}$*

Definition. B is non-degenerate if and only if the pair (V, B) is a symplectic vector space. Then e_i's and f_j's are called a Darboux basis of V.

Let B be non-degenerate and U a vector subspace of V
Remark:
$\dim U^\perp = 2n - \dim V$ and we have the following 3 scenarios.

1. U isotropic $\Leftrightarrow U^\perp \supset U$. This implies that $\dim U \leq n$
2. U Lagrangian $\Leftrightarrow U^\perp = U$. This implies $\dim U = n$.
3. U symplectic $\Leftrightarrow U^\perp \cap U = \emptyset$. This implies that $U^\perp$ is symplectic and $B|_U$ and $B|_{U^\perp}$ are non-degenerate.

Let $V = V^m$ be a vector space over $\mathbb{R}$ we have

$$\mathrm{Alt}^2(V) \cong \Lambda^2(V^*)$$

is a canonical identification. Let $v_1, \dots, v_m$ be a basis of v, then

$$\mathrm{Alt}^2(V) \ni B \mapsto \frac{1}{2} \sum B(v_i, v_j) v_i^* \wedge v_j^*$$

and the inverse $\Lambda^2(V^*) \ni \omega \mapsto B_\omega \in \mathrm{Alt}^2(V)$ is given by

$$B(v, w) = i_W(i_V \omega)$$

Suppose $m = 2n$.

Theorem. *$B \in \mathrm{Alt}^2(V)$ is non-degenerate if $\omega_B \in \Lambda^2(V)$ satisfies $\omega_B^n \neq 0$*

1/2 of Proof. B non-degenerate, let $e_1, \dots, f_n$ be a Darboux basis of V then

$$\omega_B = \sum e_i^* \wedge f_j^*$$

and we can show

$$\omega_B^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0$$

□

Notation. $\omega \in \Lambda^2(V^*)$, symplectic geometers just say "$B_\omega(v, w) = \omega(v, w)$".

Kaehler spaces

$V = V^{2n}$, V a vector space over R, $B \in \mathrm{Alt}^2(V)$ is non-generate. Assume we have another piece of structure a map $J : V \to V$ that is $\mathbb{R}$-linear and $J^2 = -I$.

Definition. B and J are **compatible** if $B(v, w) = B(Jv, Jw)$.

Exercise(not to be handed in) Let $Q(v, w) = B(v, Jw)$ show that B and J are compatible if and only if Q is symmetric.

From J we can make V a vector space over $\mathbb{C}$ by setting $\sqrt{-1}v = Jv$. So this gives V a structure of complex n-dimensional vector space.

Definition. Take the bilinear form $H : V \times V \to \mathbb{C}$ by

$$H(v, w) = \frac{1}{\sqrt{-1}}(B(v, w) + \sqrt{-1}Q(v, w))$$

B and J are compatible if and only if H is hermitian on the complex vector space V. Note that $H(v, v) = Q(v, v)$.

Definition. V, J, B is Kahler if either H is positive definite or Q is positive definite (these two are equivalent).

Consider $V^* \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, so if $l \in V^* \otimes \mathbb{C}$ then $l : V \to \mathbb{C}$.

Definition. $l \in (V^*)^{1,0}$ if it is $\mathbb{C}$-linear, i.e. $l(Jv) = \sqrt{-1}l(v)$. And $l \in (V^*)^{0,1}$ if it is $\mathbb{C}$-antilinear, i.e. $l(Jv) = -\sqrt{-1}l(v)$.

Definition. $\bar{l}v = \overline{l(v)}$. $J^*l(v) = lJ(v)$.

Then if $l \in (V^*)^{1,0}$ then $\bar{l} \in (V^*)^{0,1}$. If $l \in (V^*)^{1,0}$ then $J^*l = \sqrt{-1}l$, $l \in (V^*)^{0,1}$, $J^*l = -\sqrt{-1}l$.

So we can decompose $V^* \otimes \mathbb{C} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ i.e. decomposing into $\pm\sqrt{-1}$ eigenspace of J^* and $(V^*)^{0,1} = \overline{(V^*)^{0,1}}$.

This decomposition gives a decomposition of the exterior algebra, $\Lambda^r(V^* \otimes \mathbb{C}) = \Lambda^r(V^*) \otimes \mathbb{C}$. Now, this decomposes into bigraded pieces

$$\Lambda^r(V^* \otimes \mathbb{C}) = \bigoplus_{k+l=r} \Lambda^{k,l}(V^*)$$

$\Lambda^{k,l}(V^*)$ is the linear span of k, l forms of the form

$$\mu_1 \wedge \cdots \wedge \mu_k \wedge \bar{\nu}_1 \wedge \cdots \wedge \bar{\nu}_l \qquad \mu_i \nu_j \in (V^*)^{1,0}$$

Note that $J^* : V^* \otimes \mathbb{C} \to V^* \otimes \mathbb{C}$ can be extended to a map $J^* : \Lambda^r(V^* \otimes \mathbb{C}) \to \Lambda^r(V^* \otimes \mathbb{C})$ by setting

$$J^*(l_1 \wedge \cdots \wedge l_r) = J^*l_1 \wedge \cdots \wedge J^*l_r$$

on decomposable elements $l_1 \wedge \cdots \wedge l_r \in \Lambda^r$.

We can define complex conjugation on $\Lambda^r(V^* \otimes \mathbb{C})$ on decomposable elements $\omega = l_1 \wedge \cdots \wedge l_r$ by $\bar{\omega} = \bar{l}_1 \wedge \cdots \wedge \bar{l}_r$.

$\Lambda^r(V^* \otimes \mathbb{C}) = \Lambda^r(V) \otimes \mathbb{C}$, then $\bar{\omega} = \omega$ if and only if $\omega \in \Lambda^r(V^*)$. And if $\omega \in \Lambda^{k,l}(V^*)$ then $\bar{\omega} \in \Lambda^{l,k}(V^*)$

Proposition. *On* $\Lambda^{k,l}(V^*)$ *we have* $J^* = (\sqrt{-1})^{k-l}\,\mathrm{Id}$.

Proof. Take $\omega = \mu_1 \wedge \cdots \wedge \mu_k \wedge \bar{\nu}_1 \wedge \cdots \wedge \bar{\nu}_l$, $\mu_i, \nu_i \in (V^*)^{1,0}$ then

$$J^*\omega = J^*\mu_1 \wedge \cdots \wedge J^*\mu_k \wedge J^*\bar{\nu}_1 \wedge \cdots \wedge J^*\bar{\nu}_l = (-1)^k(-\sqrt{-1})^l\omega$$

□

Notice that for the following decomposition of $\Lambda^2(V \otimes \mathbb{C})$ the eigenvalues of J^* are given below

$$\underbrace{\Lambda^2(V \otimes \mathbb{C})}_{J^*} = \underbrace{\Lambda^{2,0}}_{1} \oplus \underbrace{\Lambda^{1,1}}_{-1} \oplus \underbrace{\Lambda^{0,2}}_{-1}$$

So if $\omega \in \Lambda^*(V^* \otimes \mathbb{C})$ then if $J\omega = \omega$.

Now, back to serious Kahler stuff.

Let V, B, J be Kahler. $B \mapsto \omega_B \in \Lambda^2(V^*) \subset \Lambda^2(V^*) \otimes \mathbb{C}$.

B is J invariant, so ω_B is J-invariant, which happens if and only if $\omega_B \in \Lambda^{1,1}(V^*)$ and ω_B is real if and only if $\bar{\omega}_B = \omega_B$.

So there is a -1 correspondence between J invariant elements of $\Lambda^2(V)$ and elements $\omega \in \Lambda^{1,1}(V^*)$ which are real.

Observe: $(V^*)^{1,0} \otimes V^*)^{0,1} \xrightarrow{\rho} \Lambda^{1,1}(V^*)$ by $\mu \otimes \nu \mapsto \mu \wedge \nu$. Let $\mu_1, \ldots, \mu_n$ be a basis of $(V^*)^{1,0}$. Take

$$\alpha = \sum a_{ij}\mu_i \otimes \bar{\mu}_j \in (V^*)^{1,0} \otimes (V^*)^{0,1}$$

Take

$$\rho(\alpha) = \sum a_{ij}\mu_i \wedge \bar{\mu}_j$$

is it true that $\overline{\rho(\alpha)} = \rho(\alpha)$. No, not always. This happens if $a_{ij} = -\overline{a_{ij}}$, equivalently $\frac{1}{\sqrt{-1}}[a_{ij}]$ is Hermitian. We have

$$\mathrm{Alt}^2(V) \ni B \mapsto \omega = \omega_B \in \Lambda^{1,1}(V^*)$$

Take $\alpha = \rho^{-1}(\omega)$, $H = \frac{1}{\sqrt{-1}}\alpha$. Then H is Hermitian.

Check that $H = \frac{1}{\sqrt{-1}}(B + \sqrt{-1}Q)$, B Kahler iff and only if H is positive definite.

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