Lecture 13

X^{2n} a real C^∞ manifold. Have $\omega \in \Omega^2(X)$, with ω closed.

For $p \in X$ we saw last time that $\Lambda^2(T_p^*) \cong \mathrm{Alt}^2(T_p)$, so $\omega_p \leftrightarrow B_p$.

Definition. ω is symplectic if for every point p, B_p is non-degenerate.

Remark: Alternatively ω is symplectic if and only if ω^n is a volume form. i.e. $\omega_p^n \neq 0$ for all p.

Theorem (Darboux Theorem). *If ω is symplectic then for every $p \in X$ there exists a coordinate patch $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on U*

$$\omega = \sum dx_i \wedge dy_i$$

(in Anna Cannas notes)

Suppose X^{2n} is a complex n-dimensional manifold. Then for $p \in X$, T_pX is a complex n-dimensional vector space. So there exists an $\mathbb{R}$-linear map $J_p : T_p \to T_p$, $J_p v = \sqrt{-1}v$ with $J_p^2 = -I$.

Definition. ω symplectic is Kahler if for every $p \in X$, B_p and J_p are compatible and the quadratic form

$$Q_p(v, w) = B_p(v, J_p w)$$

is positive definite.

This Q_p is a positive definite symmetric bilinear form on T_p for all p, so X is a Riemannian manifold as well.

We saw earlier that J_p and B_p are compatible is equivalent to the assumption that $\omega \in \Lambda^{1,1}(T_p^*)$.

Last time we say there was a mapping

$$\rho : (T^*)^{1,0} \otimes (T^*)^{0,1} \xrightarrow{\cong} \Lambda^{1,1}(T_p^*) \qquad H_p \leftrightarrow \omega_p$$

The condition $\bar{\omega}_p = \omega_p$ tells us that H_p is a hermitian bilinear form on T_p. The condition that Q_p is positive definite implies that H_p is positive definite.

Let $(U, z_1, \ldots, z_n)$ be a coordinate patch on X

$$\omega = \sqrt{-1} \sum h_{ij} dz_i \wedge d\bar{z}_j \qquad h_{i,j} \in C^\infty(U)$$

so

$$H_p = \sum h_{ij}(p)(dz_i)_p \otimes (d\bar{z}_j)_p$$

the condition that $H_p \gg 0$ ($\gg$ means positive definite) implies that $h_{ij}(p) \gg 0$.

What about the Riemannian structure? The Riemannian arc-length on U is given by

$$ds^2 = \sum h_{ij} dz_i d\bar{z}_j$$

Darboux Theorem for Kahler Manifolds

Let $(U, z_1, \ldots, z_n)$ be a coordinate patch on X, let U be biholomorphic to a polydisk $|z_1| < \epsilon_1, \ldots, |z_n| < \epsilon_n$. Let $\omega \in \Omega^{1,1}(U)$, $d\omega = 0$ be a Kaehler form. $d\omega = 0$ implies that $\overline{\partial}\omega = \partial\omega = 0$, which implies (by a theorem we proved earlier) that for some F

$$\omega = \sqrt{-1}\partial\overline{\partial}F \qquad F \in C^\infty(U)$$

(it followed from the exactness of the Dolbeault complex). Also, since $\overline{\omega} = \omega$ we get that

$$\omega = \overline{\omega} = -\sqrt{-1}\overline{\partial}\partial F = \sqrt{-1}\partial\overline{\partial}\overline{F}$$

So replacing F by $\frac{1}{2}(F + \overline{F})$ we can assume that F is real-valued. Moreover

$$\omega = \sqrt{-1}\partial\overline{\partial}F = \sqrt{-1}\sum \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

so we conclude that

$$\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}(p) \gg 0$$

for all $p \in U$, i.e. $F \in C^\infty(U)$is a strictly plurisubharmonic function.

So we've proved

Theorem (Darboux). *If ω is a Kahler form then for every poiont $p \in X$ there exists a coordinate patch $(U, z_1, \ldots, z_n)$ cenetered at p and a strictly plurisubharmonic function F on U such that on U, $\omega = \sqrt{-1}\partial\overline{\partial}F$.*

All of the local structure is locally encoded in F, the symplectic form, the Kahler form etc.

Definition. F is called the **potential function**

This function is not unique, but how not-unique is it?

Let U be a simply connected open subset of X and let $F_1, F_2 \in C^\infty(U)$ be potential functions for the Kahler metric. Let $G = F_1 - F_2$. If $\partial\overline{\partial}F_1 = \partial\overline{\partial}F_2$ then $\partial\overline{\partial}G = 0$. Now, $\partial\overline{\partial}G = 0$ implies that $d\overline{\partial}G = 0$, so $\overline{\partial}G$ is a closed 1-form. U simply connected implies that there exists an $H \in C^\infty(U)$ so that $\overline{\partial}G = dH$, so $\overline{\partial}G = \overline{\partial}H$, and $\partial H = 0$.

Let $K_1 = G - H$, $K_2 = \overline{H}$, $K_1, K_2 \in \mathcal{O}$. Ten $G = K_1 + \overline{K}_2$. But G is real-valued, so $\overline{G} = G$ so $K_1 + \overline{K}_2 = \overline{K}_1 + K_2$ which implies $K_1 - K_2 = \overline{K}_1 - \overline{K}_2$ so $K_1 - K_2$ is a real-valued holomorphic function on U. But real valued and holomorphic implies that the function is constant. Thus $K_1 - K_2$ is a constant. Adjusting this constant we get that $K_1 = K_2$.

Let $K = K_1 = K_2$, then $G = K + \overline{K}$.

Theorem. *If F_1 and F_2 are potential functions for the Kahler metric ω on U thenm $F_1 = F_2 + (K + \overline{K})$ where $K \in \mathcal{O}(U)$.*

Definition. Let X be a complex manifold, U any open subset of X. $F \in C^\infty(U)$, F is strictly plurisubharmonic if $\sqrt{-1}\partial\overline{\partial}F = \omega$ is a Kahler form on U. This is the **coordinate free definition of s.p.s.h**

Definition. An open set U of X is pseudoconvex if it admits a s.p.s.h. exhaustion function.

Remarks: U is pseudoconvex if the Dolbeault complex is exact.

Definition. X is a stein manifold if it is pseudoconvex

Examples of Kaehler Manifolds

1. $\mathbb{C}^n$. Let $F = |z|^2 = |z_1|^2 + \cdots + |z_n|^2$ and then

   $$\sqrt{-1}\partial\overline{\partial}f = \sqrt{-1}\sum dz_i \wedge d\bar{z}_j = \omega$$

   and if we say $z_i = x_i + \sqrt{-1}y$ then

   $$\omega = 2\sum dx_i \wedge dy_i$$

   then standard Darboux form.

2. Stein manifolds.

3. Complex submanifolds of Kaehler manifolds. We claim that if X^n is a complex manifold, Y^k a complex submanifold in X if $\iota : Y \to X$ is an inclusion. Then

   (a) If ω is a Kaehler form on X, $\iota^*\omega$ is a Kaehler form.

   (b) If U is an open subset of X and $F \in C^\infty(U)$ is a potential function for ω on U the ι^*F is a potential function for the form $\iota^*\omega$ on $U \cap Y$.

   b) implies *a*), so it suffices to prove *b*). Let $(U, z_1, \ldots, z_n)$ be a coordinate chart adapted for Y, i.e $Y \cap U$ is defined by $z_{k+1} = \cdots = z_n = 0$. $\omega = \sqrt{-1}\partial\overline{\partial}F$ on U, so since ι is holomorphic it commutes with ∂, $\overline{\partial}$. Then

   $$\iota^*\omega = \sqrt{-1}\partial\overline{\partial}\iota^*F \qquad \iota^*F = F(z_1, \ldots, z_k, 0, \ldots, 0)$$

   To see this is Kaehler we need only check that ι^*F is s.p.s.h. Take $p \in U \cap Y$. We consider the matrix

   $$\left[\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}(p)\right] \qquad 1 \le i, j \le k$$

   But this is the principle $k \times k$ minor of

   $$\left[\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}(p)\right] \qquad 1 \le i, j \le n$$

   and the last matrix is positive definite, by definition (and since its a hermitian matrix its principle $k \times k$ minors are positive definite)

4. All non-singular affine algebraic varieties.