Lecture 14

We discussed the Kaehler metric corresponding to the potential function $F(z) = |z|^2 = |z_1|^2 + \cdots + |z_n|^2$. Another interesting case is to take the potential function $F = \operatorname{Log}|z|^2$ on $\mathbb{C}^{n+1} - \{0\}$. This is not s.p.s.h.

But recall we have a mapping

$$\mathbb{C}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{C}P^n \qquad \pi(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$$

Theorem. *There exists a unique Kaehler form ω on $\mathbb{C}P^b$ such that $\pi^*\omega = \sqrt{-1}\partial\overline{\partial}\operatorname{Log}|z^2|$. This is called the **Fubini-Study** symplectic form.*

We'll prove this over the next few paragraphs. Let $U_i = \{[z_0, \ldots, z_n], z_i \neq 0\}$ and let $O_i = \pi^{-1}(U_i) = \{(z_0, \ldots, z_n), z_i \neq 0\}$. Define $\gamma_i : U_i \to O_i$ by mapping $\gamma_i([z_0, \ldots, z_n]) = (z_0, \ldots, z_n)/z_i$. Notice that $\pi \circ \gamma_i = \mathrm{id}_{U_i}$ and $\gamma_i \circ \pi(z_0, \ldots, z_n) = (z_0, \ldots, z_n)/z_i$.

Lemma. *Let $\mu = \sqrt{-1}\partial\overline{\partial}\operatorname{Log}|z|^2$ on $\mathbb{C}^{n+1} - \{0\}$. Then on O_i we have $\pi^*\gamma_i^*\mu = \mu$.*

Proof.

$$\pi^*\gamma_i^*\operatorname{Log}|z|^2 = (\gamma_i\pi)^*\operatorname{Log}|z|^2 = \operatorname{Log}\left(\frac{|z|^2}{|z_i|^2}\right) = \operatorname{Log}|z|^2 - \operatorname{Log}|z_i|^2$$

$$\pi^*\gamma_i^*\mu = \sqrt{-1}\pi^*\gamma_i^*\partial\overline{\partial}\operatorname{Log}|z|^2 = \sqrt{-1}\partial\overline{\partial}(\operatorname{Log}|z|^2 - \operatorname{Log}|z_i|^2)$$
$$= \sqrt{-1}\partial\overline{\partial}(\operatorname{Log}|z|^2 - \operatorname{Log} z_i - \operatorname{Log}\bar{z}_j) = \sqrt{-1}\partial\overline{\partial}\operatorname{Log}|z|^2 = \mu$$

□

Corollary. *We have local existence and uniqueness of ω on each U_i, which implies global existence and uniqueness.*

So we know there exists ω on $\mathbb{C}P^n$ such that $\pi^*\omega = \sqrt{-1}\partial\overline{\partial}\operatorname{Log}|z|^2$. We want to show that Kaehlerity of ω. Define

$$\rho_i : \mathbb{C}^n \to O_i \qquad \rho_i(z_1, \ldots, z_n) = (z_1, \ldots, 1, \ldots, z_n)$$

Then $\pi \circ \rho_i : \mathbb{C}^n \to U_i$ is a biholomorphism. It suffices to check that

$$(\pi \circ \rho_i)^*\omega = \rho_i^*\pi^*\omega = \rho^*\mu = \rho_i^*(\sqrt{-1}\partial\overline{\partial}\operatorname{Log}|z|^2)$$
$$= \sqrt{-1}\partial\overline{\partial}\operatorname{Log}(1 + |z_i|^2 + \cdots + |z_n|^2) = \sqrt{-1}\partial\overline{\partial}\operatorname{Log}(1 + |z|^2)$$

We must check that $\operatorname{Log}(1 + |z|^2)$ is s.p.s.h.

$$\frac{\partial}{\partial\bar{z}_j}\operatorname{Log}(1 + |z|^2) = \frac{z_j}{1 + |z|^2}$$
$$\frac{\partial}{\partial z_i}\partial\partial\bar{z}_j\operatorname{Log}(1 + |z|^2) = \frac{\delta_{ij}}{1 + |z|^2} - \frac{\bar{z}_i z_j}{(1 + |z|^2)^2} = \frac{1}{1 + |z|^2}((1 + |z|^2\delta_{ij} - z_j\bar{z}_i)$$

We have to check that the term in parentheses is positive, but thats not too hard.

Corollary. *All complex submanifolds of $\mathbb{C}P^n$ are Kaehler.*

Suppose we have (X, ω) a Kaehler manifold. We can associate to $\omega \in \Omega^{1,1}(X)$ another closed 2-form $\mu \in \Omega^{1,1}(X)$ called the **Ricci form**

Let $(U, z_1, \ldots, z_n)$ be a coordinate patch. Let $F \in C^\infty(U)$ be a potential function for ω on U, i.e. $\omega = \sqrt{-1}\partial\overline{\partial}F$. Let

$$G = \det\left(\frac{\partial F}{\partial z_i \partial\bar{z}_j}\right)$$

This is real and positive, so the log is well defined. Define

$$\mu = \sqrt{-1}\partial\overline{\partial}\operatorname{Log} G$$

Lemma. *μ is intrinsically defined, i.e. it is independent of F and the coordinate system*

Proof. **Independent of** F Take F_1, F_2 to be potential functions of ω on U. Then $\partial\overline{\partial}F_1 = \partial\overline{\partial}F_2$, which, in coordinates means that

$$\left[\frac{\partial F_1}{\partial z_i \partial\bar{z}_j}\right] = \left[\frac{\partial F_2}{\partial z_i \partial\bar{z}_j}\right]$$

Independent of Coordinates On $U \cap U'$ the formula's look like

$$\frac{\partial F}{\partial z_i \partial \bar{z}_j} = \sum_{k,l} \frac{\partial^2 F}{\partial z'_k \partial \bar{z}'_l} \frac{\partial z'_k}{\partial z_i} \frac{\partial \bar{z}_l}{\partial z'_j}$$

or in matrix notation

$$\left[\frac{\partial F}{\partial z_i \partial \bar{z}_j}\right] = \left[\frac{\partial z'_k}{\partial z_i}\right] \cdot \left[\frac{\partial^2 F}{\partial z'_k \partial \bar{z}'_l}\right] \cdot \left[\frac{\partial \bar{z}'_l}{\partial \bar{z}_j}\right]$$

taking determinants we get

$$\det \left[\frac{\partial F}{\partial z_i \partial \bar{z}_j}\right] = \left[\frac{\partial^2 F}{\partial z'_k \partial \bar{z}'_l}\right] H \bar{H}$$

where

$$H = \det \left[\frac{z'_k}{z_l}\right]$$

so

$$\operatorname{Log} \det \left[\frac{\partial F}{\partial z_i \partial \bar{z}_j}\right] = \operatorname{Log} \det \left[\frac{\partial^2 F}{\partial z'_i \partial \bar{z}'_j}\right] + \operatorname{Log} \det H + \operatorname{Log} \det \bar{H}$$

$\operatorname{Log} H \in \mathcal{O}(U)$ (at least on a branch). Apply $\partial\overline{\partial}$ to both sides of the above. That finishes it. □

Definition. X, ω a Kaehler manifold and μ is the Ricci form. Then X is called **Kaehler-Einstein** if there exists a constant such that $\mu = \lambda\omega$.

Take $\mu = \lambda\omega$, $\lambda \neq 0$. Let $(U, z_1, \ldots, z_n)$ be a coordinate patch. For $F \in C^\infty(U)$ a potential function for ω on U

$$\mu = \sqrt{-1}\partial\overline{\partial} \operatorname{Log} \det \left(\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}\right) = \lambda\omega = \lambda\sqrt{-1}\partial\overline{\partial} F$$

By a theorem we proved last time

$$\operatorname{Log} \det \left(\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}\right) = \lambda F = G + \overline{G} \qquad G \in \mathcal{O}(U)$$

Take F and replace it by

$$F \rightsquigarrow F + \frac{1}{\lambda}(G + \overline{G})$$

then

$$\operatorname{Log} \det \left(\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}\right) = \lambda F \qquad \boxed{\det \left(\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}\right) = e^{\lambda F}}$$

The boxed formula is the Monge-Ampere equation. This is essential an equation for constructing Einstein-Kahler metrics.

Exercise Check that the Fubini-Study potential is Kaehler-Einstein with $\lambda = -(n+1)$. $F = \operatorname{Log}(1+|z|^2)$ locally on each U_i. So we need to check that $F = \operatorname{Log}(1 + |z|^2)$ satisfies the Monge-Ampere equations.