Lecture 15

Homework problem number 2. X a complex manifold. We know we have the splitting

$$\Omega^r(X) = \bigoplus_{p+q} \Omega^{p,q}(X) \qquad d = \partial + \overline{\partial}$$

We get the Dolbeault complex $\Omega^{0,0}(X) \xrightarrow{\bar\partial} \Omega^{0,1}(X) \xrightarrow{\bar\partial} \ldots$ and for every p we get a generalized Dolbeault complex

$$\Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \Omega^{p,2}(X) \xrightarrow{\overline{\partial}} \cdots$$

this is the p-Dolbeault complex. Take $\ker \overline{\partial} : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ this is $\mathcal{O}(X)$ and in general $\ker \overline{\partial} : \Omega^{p,0}(X) \to \Omega^{p,1}(X)$. Call this $A^p(X)$. For $\mu \in A^p(X)$ pick a coordinate patch $(U, z_1, \ldots, z_n)$ then

$$\mu = \sum f_I(z) dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

and $\overline{\partial}\mu = 0$ implies that $\overline{\partial} f_I = 0$, so $f_I \in \mathcal{O}(U)$. These A^p are called the holomorphic de Rham complex.

More general, take U open in X. Then $\mathcal{A}^p(X)$ defines a sheaf $\mathcal{A}^p$ on X.

Exercise Let $U = \{U_i, i \in I\}$ be a cover of X by pseudoconvex open sets. Show that the Cech cohomology group $H^q(U, \mathcal{A}^p)$ coincide with the cohomology groups of

$$\Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \Omega^{p,2}(X) \xrightarrow{\overline{\partial}} \cdots$$

We did the special case $p = 0$, i.e. we showed $H^q(U, \mathcal{O}) \cong$ the Dolbeault complex.

The idea is to reduce this to the following exercise in diagram chasing. Let $C = \bigoplus C^{i,j}$ be a bigraded vector space with commuting coboundary operators $\delta : C^{i,j} \to C^{i+1,j}$ and $d : C^{i,j} \to C^{i,j+1}$.

Let $V_i = \ker d_i : C^{i,0} \to C^{i,1}$. Note that since $d\delta = \delta d$ that $\delta V_i \subset V_{i+1}$. Also let $W = \ker \delta_i : C^{0,i} \to C^{1,i}$ and $dW_i \subset W_{i+1}$.

Theorem. *Suppose that the sequence*

$$C^{0,i} \xrightarrow{\delta} C^{1,i} \xrightarrow{\delta} C^{2,i} \xrightarrow{\delta} \cdots$$

and the sequence

$$C^{i,0} \xrightarrow{d} C^{i,1} \xrightarrow{d} C^{i,2} \xrightarrow{d} \cdots$$

are exact for all i. Prove that the cohomology groups of

$$0 \longrightarrow V_0 \xrightarrow{\delta} V_1 \xrightarrow{\delta} V_2 7 \xrightarrow{\delta} \cdots$$

and

$$0 \longrightarrow W_0 \xrightarrow{d} W_1 \xrightarrow{d} W_2 \xrightarrow{d} \cdots$$

are isomorphic.