Lecture 17

4.3 Smoothing operators

Let X be an n-dimensional manifold equipped with a smooth non-vanishing measure, dx. Given $K \in \mathcal{C}^\infty(X \times X)$, one can define an operator

$$T_K : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

by setting

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$$T_K f(x) = \int K(x,y) f(y)\, dy\,. \tag{4.3.1}$$

Operators of this type are called *smoothing* operators. The definition (4.3.1) involves the cho ice of the measure, dx, however, it's easy to see that the notion of "smoothing operator" doesn't depend on this choice. Any other smooth measure will be of the form, $\varphi(x)\, dx$, where φ is an everywhere-positive $\mathcal{C}^\infty$ function, and if we replace dy by $\varphi(y)\, dy$ in (4.3.1) we get the smoothing operator, T_{K_1}, where $K_1(x,y) = K(x,y)\,\varphi(y)$.

A couple of elementary remarks about smoothing operators:

1. Let $L(x,y) = \overline{K(y,x)}$. Then T_L is the *transpose* of T_K. For f and g in $\mathcal{C}_0^\infty(X)$,

$$\begin{aligned} \langle T_K f, g\rangle &= \int \overline{g}(x)\left(\int K(x,y)f(y)\,dy\right)dx \\ &= \int f(y)\overline{(T_L g)(y)\,dy} = \langle f, T_L g\rangle\,. \end{aligned}$$

2. If X is compact, the composition of two smoothing operators is a smoothing operator. Explicitly:

$$T_{K_1}T_{K_2} = T_{K_3}$$

where

$$K_3(x,y) = \int K_1(x,z)K_2(z,y)\,dz\,.$$

We will now give a rough outline of how our proof of Theorem 4.2 will go. Let $I : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ be the identity operator. We will prove in the next few sections the following two results.

Theorem. *The elliptic operator, P is right-invertible modulo smoothing operators, i.e., there exists an operator, $Q : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ and a smoothing operator, T_K, such that*

$$PQ = I - T_K \tag{4.3.2}$$

and

Theorem. *The Fredholm theorem is true for the operator, $I - T_K$, i.e., the kernel of this operator is finite dimensional, and $f \in \mathcal{C}^\infty(X)$ is in the image of this operator if and only if it is orthogonal to kernel of the operator, $I - T_L$, where $L(x,y) = \overline{K(y,x)}$.*

Remark. In particular since T_K is the transpose of T_L, the kernel of $I - T_L$ is finite dimensional.

The proof of Theorem 4.3 is very easy, and in fact we'll leave it as a series of exercises. (See §**??**.) The proof of Theorem 4.3, however, is a lot harder and will involve the theory of pseudodifferential operators on the n-torus, T^n.

We will conclude this section by showing how to deduce Theorem 4.2 from Theorems 4.3 and 4.3. Let V be the kernel of $I - T_L$. By Theorem 4.3, V is a finite dimensional space, so every element, f, of $\mathcal{C}^\infty(X)$ can be written uniquely as a sum

$$f = g + h \tag{4.3.3}$$

where g is in V and h is orthogonal to V. Indeed, if $f_1, \ldots, f_m$ is an orthonormal basis of V with respect to the L^2 norm

$$g = \sum \langle f, f_i\rangle f_i$$

and $h = f - g$. Now let U be the orthocomplement of $V \cap \text{Image } P$ in V.

Proposition. *Every $f \in \mathcal{C}^\infty(M)$ can be written uniquely as a sum*

$$f = f_1 + f_2 \tag{4.3.4}$$

where $f_1 \in U$, $f_2 \in \text{Image } P$ and f_1 is orthogonal to f_2.

Proof. By Theorem 4.3

$$\text{Image } P \subset \text{Image}\,(I - T_K)\,. \tag{4.3.5}$$

Let g and h be the "g" and "h" in (4.3.3). Then since h is orthogonal to V, it is in $\text{Image}\,(I - T_K)$ by Theorem 4.3 and hence in Image P by (4.3.5). Now let $g = f_1 + g_2$ where f_1 is in U and g_2 is in the orthocomplement of U in V (i.e., in $V \cap \text{Image } P$). Then

$$f = f_1 + f_2$$

where $f_2 = g_2 + h$ is in Image P. Since f_1 is orthogonal to g_2 and h it is orthogonal to f_2.

□

Next we'll show that

$$U = \text{Ker } P^t\,. \tag{4.3.6}$$

Indeed $f \in U \Leftrightarrow f \perp \text{Image } P \Leftrightarrow \langle f, Pu\rangle = 0$ for all $u \Leftrightarrow \langle P^t f, u\rangle = 0$ for all $u \leftrightarrow P^t f = 0$.

This proves that all the assertions of Theorem 4.3 are true except for the finite dimensionality of Ker P. However, (4.3.6) tells us that Ker P^t is finite dimensional and so, with P and P^t interchanged, Ker P is finite dimensional.

4.4 Fourier analysis on the n-torus

In these notes the "n-torus" will be, by definition, the manifold: $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. A $\mathcal{C}^\infty$ function, f, on T^n can be viewed as a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$ which is *periodic* of period 2π: For all $k \in \mathbb{Z}^n$

$$f(x + 2\pi k) = f(x)\,. \tag{4.4.1}$$

Basic examples of such functions are the functions

$$e^{ikx}\,, \quad k \in \mathbb{Z}^n\,, \quad kx = k_1x_1 + \cdots k_nx_n\,.$$

Let $\mathcal{P} = \mathcal{C}^\infty(T^n) = \mathcal{C}^\infty$ functions on $\mathbb{R}^n$ satisfying (4.4.1), and let $Q \subseteq \mathbb{R}^n$ be the open cube

$$0 < x_i < 2\pi\,. \quad i = 1, \ldots, n\,.$$

Given $f \in \mathcal{P}$ we'll define

$$\int_{T^n} f\,dx = \left(\frac{1}{2\pi}\right)^n \int_Q f\,dx$$

and given $f, g \in \mathcal{P}$ we'll define their L^2 inner product by

$$\langle f, g\rangle = \int_{T^n} f\overline{g}\,dx\,.$$

I'll leave you to check that

$$\langle e^{ikx}\,,\, e^{i\ell x}\rangle$$

is zero if $k \neq \ell$ and 1 if $k = \ell$. Given $f \in \mathcal{P}$ we'll define the k^{th} *Fourier coefficient* of f to be the L^2 inner product

$$c_k = c_k(f) = \langle f, e^{ikx}\rangle = \int_{T^n} fe^{-ikx}\,dx\,.$$

The *Fourier series* of f is the formal sum

$$\sum c_k e^{ikx}\,, \quad k \in \mathbb{Z}^n\,. \tag{4.4.2}$$

In this section I'll review (very quickly) standard facts about Fourier series.
It's clear that $f \in \mathcal{P} \Rightarrow D^\alpha f \in \mathcal{P}$ for all multi-indices, α.

Proposition. *If $g = S^{\alpha f}$*

$$c_k(g) = k^\alpha c_k(f)\,.$$

Proof.

$$\int_{T^n} D^\alpha f e^{-ikx}\,dx = \int_{T^n} f\overline{D^\alpha e^{ikx}}\,dx\,.$$

Now check

$$D^\alpha e^{ikx} = k^\alpha e^{ikx}\,.$$

□

Corollary. *For every integer $r > 0$ there exists a constant C_r such that*

$$|c_k(f)| \leq C_r(1 + |k|^2)^{-r/2}\,. \tag{4.4.3}$$

Proof. Clearly

$$|c_k(f)| \leq \frac{1}{(2\pi)^n} \int_{T^n} |f|\,dx = C_0\,.$$

Moreover, by the result above, with $g = D^\alpha f$

$$k^\alpha |C_K(f)| = |C_K(g)| \leq C_\alpha$$

and from this it's easy to deduce an estimate of the form (4.4.3).

□

Proposition. *The Fourier series (4.4.2) converges and this sum is a $\mathcal{C}^\infty$ function.*

To prove this we'll need

Lemma. *If $m > n$ the sum*

$$\sum \left(\frac{1}{1+|k|^2}\right)^{m/2} , \quad k \in \mathbb{Z}^n , \tag{4.4.4}$$

converges.

Proof. By the "integral test" it suffices to show that the integral

$$\int_{\mathbb{R}^n} \left(\frac{1}{1+|x|^2}\right)^{m/2} dx$$

converges. However in polar coordinates this integral is equal to

$$\gamma_{n-1} \int_0^\infty \left(\frac{1}{1+|r|^2}\right)^{m/2} r^{n-1}\, dr$$

(γ_{n-1} being the volume of the unit $n-1$ sphere) and this converges if $m > n$. □

Combining this lemma with the estimate (4.4.3) one sees that (4.4.2) converges absolutely, i.e.,

$$\sum |c_k(f)|$$

converges, and hence (4.4.2) converges uniformly to a continuous limit. Moreover if we differentiate (4.4.2) term by term we get

$$D^\alpha \sum c_k e^{ikx} = \sum k^\alpha c_k e^{ikx}$$

and by the estimate (4.4.3) this converges absolutely and uniformly. Thus the sum (4.4.2) exists, and so do its derivatives of all orders.

Let's now prove the fundamental theorem in this subject, the identity

$$\sum c_k(f) e^{ikx} = f(x) . \tag{4.4.5}$$

Proof. Let $\mathcal{A} \subseteq \mathcal{P}$ be the algebra of trigonometric polynomials:

$$f \in \mathcal{A} \Leftrightarrow f = \sum_{|k| \leq m} a_k e^{ikx}$$

for some m. □

Claim. This is an algebra of continuous functions on T^n having the Stone–Weierstrass properties

1) Reality: If $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$.
2) $1 \in \mathcal{A}$.
3) If x and y are points on T^n with $x \neq y$, there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

Proof. Item 2 is obvious and item 1 follows from the fact that $\overline{e^{ikx}} = e^{-ikx}$. Finally to verify item 3 we note that the finite set, $\{e^{ix_1}, \ldots, e^{ix_n}\}$, already separates points. Indeed, the map

$$T^n \to (S^1)^n$$

mapping x to $e^{ix_1}, \ldots, e^{ix_n}$ is bijective.

Therefore by the Stone–Weierstrass theorem $\mathcal{A}$ is dense in $C^0(T^n)$. Now let $f \in \mathcal{P}$ and let g be the Fourier series (4.4.2). Is f equal to g? Let $h = f - g$. Then

$$\begin{aligned} \langle h, e^{ikx} \rangle &= \langle f, e^{ikx} \rangle - \langle g, e^{ikx} \rangle \\ &= c_k(f) - c_k(f) = 0 \end{aligned}$$

so $\langle h, e^{ikx} \rangle = 0$ for all e^{ikx}, hence $\langle h, \varphi \rangle = 0$ for all $\varphi \in \mathcal{A}$. Therefore since $\mathcal{A}$ is dense in $\mathcal{P}$, $\langle h, \varphi \rangle = 0$ for all $\varphi \in \mathcal{P}$. In particular, $\langle h, h \rangle = 0$, so $h = 0$.

□

I'll conclude this review of the Fourier analysis on the n-torus by making a few comments about the L^2 theory.

The space, $\mathcal{A}$, is dense in the space of continuous functions on T^n and this space is dense in the space of L^2 functions on T^n. Hence if $h \in L^2(T^n)$ and $\langle h, e^{ikx} \rangle = 0$ for all k the same argument as that I sketched above shows that $h = 0$. Thus

$$\{e^{ikx} \,, \, k \in \mathbb{Z}^n\}$$

is an orthonormal basis of $L^2(T^n)$. In particular, for every $f \in L^2(T^n)$ let

$$c_k(f) = \langle f, e^{ikx} \rangle \, .$$

Then the Fourier series of f

$$\sum c_k(f) e^{ikx}$$

converges in the L^2 sense to f and one has the Plancherel formula

$$\langle f, f \rangle = \sum |c_k(f)|^2 \, , \quad k \in \mathbb{Z}^n \, .$$