Lecture 18

4.5 Pseudodifferential operators on T^n

In this section we will prove Theorem 4.2 for elliptic operators on T^n. Here's a road map to help you navigate this section. §4.5.1 is a succinct summary of the material in §4. Sections 4.5.2, 4.5.3 and 4.5.4 are a brief account of the theory of pseudodifferential operators on T^n and the symbolic calculus that's involved in this theory. In §4.5.5 and 4.5.6 we prove that an elliptic operator on T^n is right invertible modulo smoothing operators (and that its inverse is a pseudodifferential operator). Finally, in §4.5.7, we prove that pseudodifferential operators have a property called "pseudolocality" which makes them behave in some ways like differential operators (and which will enable us to extend the results of this section from T^n to arbitrary compact manifolds).

Some notation which will be useful below: for $a \in \mathbb{R}^n$ let

$$\langle a \rangle = (|a|^2 + 1)^{\frac{1}{2}} .$$

Thus

$$|a| \leq \langle a \rangle$$

and for $|a| \geq 1$

$$\langle a \rangle \leq 2|a| .$$

4.5.1 The Fourier inversion formula

Given $f \in \mathcal{C}^\infty(T^n)$, let $c_k(f) = \langle f, e^{ikx} \rangle$. Then:

1) $c_k(D^{\alpha f}) = k^\alpha c_k(f)$.

2) $|c_k(f)| \leq C_r \langle k \rangle^{-r}$ for all r.

3) $\sum c_k(f) e^{ikx} = f$.

Let S be the space of functions,

$$g : \mathbb{Z}^n \to \mathbb{C}$$

satisfying

$$|g(k)| \leq C_r \langle k \rangle^{-r}$$

for all r. Then the map

$$F : \mathcal{C}^\infty(T^n) \to S\,, \quad Ff(k) = c_k(f)$$

is bijective and its inverse is the map,

$$g \in S \to \sum g(k) e^{ikx}\,.$$

4.5.2 Symbols

A function $a : T^n \times \mathbb{R}^n \to \mathbb{C}$ is an $\mathcal{S}^m$ if, for all multi-indices, α and β,

$$|D_x^\alpha D_\xi^\beta| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}\,. \tag{5.2.1}$$

Examples

1) $a(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$, $a_\alpha \in \mathcal{C}^\infty(T^n)$.

2) $\langle \xi \rangle^m$.

3) $a \in \mathcal{S}^\ell$ and $b \in \mathcal{S}^m \Rightarrow ab \in S^{\ell+m}$.

4) $a \in \mathcal{S}^m \Rightarrow D_x^\alpha D_\xi^\beta a \in \mathcal{S}^{m-|\beta|}$.

The asymptotic summation theorem

Given $b_i \in \mathcal{S}^{m-i}$, $i = 0, 1, \ldots$, there exists a $b \in \mathcal{S}^m$ such that

$$b - \sum_{j<i} b_j \in \mathcal{S}^{m-i}. \tag{5.2.2}$$

Proof. Step 1. Let $\ell = m + \epsilon$, $\epsilon > 0$. Then

$$|b_i(x,\xi)| < C_i \langle \xi \rangle^{m-i} = \frac{c_i \langle \xi \rangle^{\ell - i}}{\langle \xi \rangle^\epsilon}\,.$$

Thus, for some λ_i,

$$|b_i(x,\xi) < \frac{1}{2^i} \langle \xi \rangle^{\ell-i}$$

for $|\xi| \; > \lambda_i$. We can assume that $\lambda_i \to +\infty$ as $i \to +\infty$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be bounded between 0 and 1 and satisfy $\rho(t) = 0$ for $t < 1$ and $\rho(t) = 1$ for $t > 2$. Let

$$b = \sum \rho\left(\frac{|\xi|}{\lambda_i}\right) b_i(x,\xi)\,. \tag{5.2.3}$$

Then b is in $\mathcal{C}^\infty(T^n \times \mathbb{R}^n)$ since, on any compact subset, only a finite number of summands are non-zero. Moreover, $b - \sum_{j<i} b_j$ is equal to:

$$\sum_{j<i} \left(\rho\left(\frac{|\xi|}{\lambda_j}\right) - 1 \right) b_j + b_i + \sum_{j>i} \rho\left(\frac{|\xi|}{\lambda_j}\right) b_j\,.$$

The first summand is compactly supported, the second summand is in $\mathcal{S}^{m-1}$ and the third summand is bounded from above by

$$\sum_{k>i} \frac{1}{2^k} \langle \xi \rangle^{\ell-k}$$

which is less than $\langle\xi\rangle^{\ell-(i+1)}$ and hence, for $\epsilon < 1$, less than $\langle\xi\rangle^{m-i}$.

Step 2. For $|\alpha| + |\beta| \le N$ choose λ_i so that

$$|D_x^\alpha D_\xi^\beta b_i(x,\xi)| \le \frac{1}{2^i}\langle\xi\rangle^{\ell-i-|\beta|}$$

for $\lambda_i < |\xi|$. Then the same argument as above implies that

$$D_x^\alpha D_\xi^\beta (b - \sum_{j,i} b_j) \le C_N \langle\xi\rangle^{m-i-|\beta|} \tag{5.2.4}$$

for $|\alpha| + |\beta| \le N$.

Step 3. The sequence of λ_i's in step 2 depends on N. To indicate this dependence let's denote this sequence by $\lambda_{i,N}$, $i = 0, 1, \ldots$. We can, by induction, assume that for all i, $\lambda_{i,N} \le \lambda_{i,N+1}$. Now apply the Cantor diagonal process to this collection of sequences, i.e., let $\lambda_i = \lambda_{i,i}$. Then b has the property (5.2.4) for all N.

We will denote the fact that b has the property (5.2.2) by writing

$$b \sim \sum b_i\,. \tag{5.2.5}$$

The symbol, b, is not unique, however, if $b \sim \sum b_i$ and $b' \sim \sum b_i$, $b - b'$ is in the intersection, $\bigcap \mathcal{S}^\ell$, $-\infty < \ell < \infty$.

□

4.5.3 Pseudodifferential operators

Given $a \in \mathcal{S}^m$ let

$$T_a^0 : S \to \mathcal{C}^\infty(T^n)$$

be the operator

$$T_a^0 g = \sum a(x,k) g(k) e^{ikx}\,.$$

Since

$$|D^\alpha a(x,k) e^{ikx}| \le C_\alpha \langle k\rangle^{m+\langle\alpha\rangle}$$

and

$$|g(k)| \le C_\alpha \langle k\rangle^{-(m+n+|\alpha|+1)}$$

this operator is well-defined, i.e., the right hand side is in $\mathcal{C}^\infty(T^n)$. Composing T_a^0 with F we get an operator

$$T_a : \mathcal{C}^\infty(T^n) \to \mathcal{C}^\infty(T^n)\,.$$

We call T_a the pseudodifferential operator with symbol a.

Note that

$$T_a e^{ikx} = a(x,k) e^{ikx}\,.$$

Also note that if

$$P = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \tag{5.3.1}$$

and

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\,. \tag{5.3.2}$$

Then

$$P = T_p\,.$$

4.5.4 The composition formula

Let P be the differential operator (5.3.1). If a is in $\mathcal{S}^r$ we will show that PT_a is a pseudodifferential operator of order $m+r$. In fact we will show that

$$PT_a = T_{p\circ a} \tag{5.4.1}$$

where

$$p\circ a(x,\xi) = \sum_{|\alpha|\leq m} \frac{1}{\beta!}\partial_\xi^\beta p(x,\xi) D_x^\beta a(x,\xi) \tag{5.4.2}$$

and $p(x,\xi)$ is the function (5.3.2).

Proof. By definition

$$\begin{aligned} PT_a e^{ikx} &= Pa(x,k)e^{ikx} \\ &= e^{ikx}(e^{-ikx}Pe^{ikx})a(x,k)\,. \end{aligned}$$

Thus PT_a is the pseudodifferential operator with symbol

$$e^{-ix\xi}Pe^{ix\xi}a(x,\xi)\,. \tag{5.4.3}$$

However, by (5.3.1):

$$\begin{aligned} e^{-ix\xi}Pe^{ix\xi}u(x) &= \sum a_\alpha(x)e^{-ix\xi}D^\alpha e^{ix\xi}u(x) \\ &= \sum a_\alpha(x)(D+\xi)^\alpha u(x) \\ &= P(x,D+\xi)u(x)\,. \end{aligned}$$

Moreover,

$$p(x,\eta+\xi) = \sum \frac{1}{\beta!}\,\frac{\partial}{\partial\xi^\beta}p(x,\xi)\eta^\beta\,,$$

so

$$p(x,D+\xi)u(x) = \sum \frac{1}{\beta!}\,\frac{\partial}{\partial\xi^\beta}p(x,\xi)D^\beta u(x)$$

and if we plug in $a(x,\xi)$ for $u(x)$ we get, by (5.4.3), the formula (5.4.2) for the symbol of PT_a.

□

4.5.5 The inversion formula

Suppose now that the operator (5.3.1) is elliptic. We will prove below the following inversion theorem.

Theorem. *There exists an $a \in \mathcal{S}^{-m}$ and an $r \in \bigcap S^\ell$, $-\infty < \ell < \infty$, such that*

$$PT_a = I - T_r\,.$$

Proof. Let

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\,.$$

By ellipticity $p_m(x,\xi) \neq 0$ for $\xi \notin 0$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a function satisfying $\rho(t)=0$ for $t<1$ and $\rho(t)=1$ for $t>2$. Then the function

$$a_0(x,\xi) = \rho(|\xi|)\frac{1}{p_m(x,\xi)} \tag{5.5.1}$$

is well-defined and belongs to S^{-m}. To prove the theorem we must prove that there exist symbols $a \in \mathcal{S}^{-m}$ and $r \in \bigcap \mathcal{S}^\ell$, $-\infty < \ell < \infty$, such that

$$p\circ q = 1 - r\,.$$

We will deduce this from the following two lemmas.

Lemma. *If $b \in \mathcal{S}^i$ then*

$$b - p \circ a_0 b$$

is in $\mathcal{S}^{i-1}$.

Proof. Let $q = p - p_m$. Then $q \in \mathcal{S}^{m-1}$ so $q \circ a_0 b$ is in $\mathcal{S}^{i-1}$ and by (5.4.2)

$$\begin{aligned} p \circ a_0 b &= p_m \circ a_0 b + q \circ a_0 b \\ &= p_m a_0 b + \cdots = b + \cdots \end{aligned}$$

where the dots are terms of order $i - 1$.

□

Lemma. *There exists a sequence of symbols $a_i \in \mathcal{S}^{-m-i}$, $i = 0, 1, \ldots$, and a sequence of symbols $r_i \in \mathcal{S}^{-i}$, $i = 0, \ldots$, such that a_0 is the symbol (5.5.1), $r_0 = 1$ and*

$$p \circ a_i = r_i - r_{i+1}$$

for all i.

Proof. Given $a_0, \ldots, a_{i-1}$ and $r_0, \ldots r_i$, let $a_i = r_i a_0$ and $r_{i+1} = r_i - p \circ a_i$. By Lemma 4.5.5, $r_{i+1} \in \mathcal{S}^{-i-1}$. □

Now let $a \in \mathcal{S}^{-m}$ be the "asymptotic sum" of the a_i's

$$a \sim \sum a_i \,.$$

Then

$$p \circ a \sim \sum p \circ a_i = \sum_{i=1}^{\infty} r_i - r_{i=1} = r_0 = 1 \,,$$

so $1 - p \circ a \sim 0$, i.e., $r = 1 - p \circ q$ is in $\bigcap \mathcal{S}^\ell$, $-\infty < \ell < \infty$.

□

4.5.6 Smoothing properties of ΨDO's

Let $a \in \mathcal{S}^\ell$, $\ell < -m - n$. We will prove in this section that the sum

$$K_a(x, y) = \sum a(x, k) e^{ik(x-y)} \tag{5.6.1}$$

is in $C^m(T^\beta \times T^n)$ and that T_a is the integral operator associated with K_a, i.e.,

$$T_a u(x) = \int K_a(x, y) u(y)\, dy \,.$$

Proof. For $|\alpha| + |\beta| \leq m$

$$D_x^\alpha D_y^\beta a(x, k) e^{ik(x-y)}$$

is bounded by $\langle k \rangle^{\ell + |\alpha| + |\beta|}$ and hence by $\langle k \rangle^{\ell + m}$. But $\ell + m < -n$, so the sum

$$\sum D_x^\alpha D_y^\beta a(x, k) e^{ik(x-y)}$$

converges absolutely. Now notice that

$$\int K_a(x, y) e^{iky}\, dy = a(x, k) e^{ikx} = T_\alpha e^{ikx} \,.$$

Hence T_a is the integral operators defined by K_a. Let

$$\mathcal{S}^{-\infty} = \bigcap \mathcal{S}^\ell \,, \quad -\infty < \ell \infty \,. \tag{5.6.2}$$

If a is in $\mathcal{S}^{-\infty}$, then by (5.6.1), T_a is a smoothing operator.

□

4.5.7 Pseudolocality

We will prove in this section that if f and g are $\mathcal{C}^\infty$ functions on T^n with non-overlapping supports and a is in $\mathcal{S}^m$, then the operator

$$u \in \mathcal{C}^\infty(T^n) \to fT_a gu \tag{5.7.1}$$

is a smoothing operator. (This property of pseudodifferential operators is called *pseudolocality.*) We will first prove:

Lemma. *If $a(x,\xi)$ is in $\mathcal{S}^m$ and $w \in \mathbb{R}^n$, the function,*

$$a_w(x,\xi) = a(x,\xi+w) - a(x,\xi) \tag{5.7.2}$$

is in S^{m-1}.

Proof. Recall that $a \in \mathcal{S}^m$ if and only if

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}\,.$$

From this estimate is is clear that if a is in $\mathcal{S}^m$, $a(x,\xi+w)$ is in $\mathcal{S}^m$ and $\frac{\partial a}{\partial \xi_i}(x,\xi)$ is in $\mathcal{S}^{m-1}$, and hence that the integral

$$a_w(x,\xi) = \int_0^1 \sum_i \frac{\partial a}{\partial \xi_i}(x,\xi+tw)\,dt$$

in $\mathcal{S}^{m-1}$.

Now let ℓ be a large positive integer and let a be in $\mathcal{S}^m$, $m < -n-\ell$. Then

$$K_a(x,y) = \sum a(x,k)e^{ik(x-y)}$$

is in $C^\ell(T^n \times T^n)$, and T_a is the integral operator defined by K_a. Now notice that for $w \in \mathbb{Z}^n$

$$(e^{-i(x-y)w}-1)K_a(x,y) = \sum a_w(x,k)e^{ik(x-y)}\,, \tag{5.7.3}$$

so by the lemma the left hand side of (5.7.3) is in $C^{\ell+1}(T^n \times T^n)$. More generally,

$$(e^{-i(x-y)w}-1)^N K_a(x,y) \tag{5.7.4}$$

is in $C^{\ell+N}(T^n \times T^n)$. In particular, if $x \neq y$, then for some $1 \le i \le n$, $x_i - y_i \not\equiv 0 \mod 2\pi Z$, so if

$$w = (0,0,\ldots,1,0,\ldots,0)\,,$$

(a "1" in the i^{th}-slot), $e^{i(x-y)w} \neq 1$ and, by (5.7.4), $K_a(x,y)$ is $C^{\ell+N}$ is a neighborhood of (x,y). Since N can be arbitrarily large we conclude

Lemma. *$K_a(x,y)$ is a $\mathcal{C}^\infty$ function on the complement of the diagonal in $T^n \times T^n$.*

Thus if f and g are $\mathcal{C}^\infty$ functions with non-overlapping support, $fT_a g$ is the smoothing operator, T_K, where

$$K(x,y) = f(x)K_a(x,y)g(y)\,. \tag{5.7.5}$$

□

We have proved that T_a is pseudolocal if $a \in \mathcal{S}^m$, $m < -n-\ell$, ℓ a large positive integer. To get rid of this assumption let $\langle D\rangle^N$ be the operator with symbol $\langle\xi\rangle^N$. If N is an even positive integer

$$\langle D\rangle^N = (\sum D_i^2 + I)^{\frac{N}{2}}$$

is a differential operator and hence is a *local* operator: if f and g have non-overlapping supports, $f\langle D\rangle^N g$ is identically zero. Now let $a_N(x,\xi) = a(x,\xi)\langle\xi\rangle^{-N}$. Since $a_N \in \mathcal{S}^{m-N}$, T_{a_N} is pseudolocal for N large. But $T_a = T_{a_N}\langle D\rangle^N$, so T_a is the composition of an operator which is pseudolocal with an operator which is local, and therefore T_a itself is pseudolocal.

4.6 Elliptic operators on open subsets of T^n

Let U be an open subset of T^n. We will denote by $\iota_U : U \to T^n$ the inclusion map and by $\iota_U^* : \mathcal{C}^\infty(T^n) \to \mathcal{C}^\infty(U)$ the restriction map: let V be an open subset of T^n containing $\overline{U}$ and

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \,, \quad a_\alpha(x) \in \mathcal{C}^\infty(V)$$

an elliptic m^{th} order differential operator. Let

$$P^t = \sum_{|\alpha| \le m} D^\alpha \overline{a}_\alpha(x)$$

be the transpose operator and

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

the symbol at P. We will prove below the following localized version of the inversion formula of § 4.5.5.

Theorem. *There exist symbols, $a \in \mathcal{S}^{-m}$ and $r \in \mathcal{S}^{-\infty}$ such that*

$$P \iota_U^* T_a = \iota_U^* (I - T_r) \,. \tag{4.6.1}$$

Proof. Let $\gamma \in \mathcal{C}_0^\infty(V)$ be a function which is bounded between 0 and 1 and is identically 1 in a neighborhood of $\overline{U}$. Let

$$Q = PP^t\gamma + (1-\gamma)(\sum D_\iota^2)^n \,.$$

This is a globally defined $2m^{\text{th}}$ order differential operator in T^n with symbol,

$$\gamma(x)|P_m(x,\xi)|^2 + (1-\gamma(x))|\xi|^{2m} \tag{4.6.2}$$

and since (4.6.2) is non-vanishing on $T^n \times (\mathbb{R}^n - 0)$, this operator is elliptic. Hence, by Theorem 4.5.5, there exist symbols $b \in \mathcal{S}^{-2m}$ and $r \in \mathcal{S}^{-\infty}$ such that

$$QT_b = I - T_r \,.$$

Let $T_a = P^t \gamma T_b$. Then since $\gamma \equiv 1$ on a neighborhood of $\overline{U}$,

$$\begin{aligned} \iota_U^*(I - T_r) &= \iota_U^* Q T_b \\ &= \iota_U^*(PP^t\gamma T_b + (1-\gamma)\sum D_i^2 T_b) \\ &= \iota_U^* PP^t \gamma T_b \\ &= P \iota_U^* P^t \gamma T_b = P \iota_U^* T_a \,. \end{aligned}$$

4.7 Elliptic operators on compact manifolds

Let X be a compact n dimensional manifold and

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

an elliptic m^{th} order differential operator. We will show in this section how to construct a *parametrix* for P: an operator

$$Q : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

such that $I - PQ$ is smoothing.

Let V_i, $i = 1, \ldots, N$ be a covering of X by coordinate patches and let U_i, $i = 1, \ldots, N$, $\overline{U}_i \subset V_i$ be an open covering which refines this covering. We can, without loss of generality, assume that V_i is an open subset of the hypercube

$$\{x \in \mathbb{R}^n \quad 0 < x_i < 2\pi \quad i = 1, \ldots, n\}$$

and hence an open subset of T^n. Let

$$\{\rho_i \in \mathcal{C}_0^\infty(U_i)\,, \quad i = 1, \ldots, N\}$$

be a partition of unity and let $\gamma_i \in \mathcal{C}_0^\infty(U_i)$ be a function which is identically one on a neighborhood of the support of ρ_i. By Theorem 4.6, there exist symbols $a_i \in \mathcal{S}^{-m}$ and $r_i \in \mathcal{S}^{-\infty}$ such that on T^n:

$$P\iota_{U_i}^* T_{a_i} = \iota_{U_i}^*(I - T_{r_i})\,. \tag{4.7.1}$$

Moreover, by pseudolocality $(1-\gamma_i)T_{a_i}\rho_i$ is smoothing, so

$$\gamma_i T_{a_i}\rho_i - \iota_{U_i}^* T_{a_i}\rho_i$$

and

$$P\gamma_i T_{a_i}\rho_i - P\iota_{U_i}^* T_{a_i}\rho_i$$

are smoothing. But by (4.7.1)

$$P\iota_{U_i}^* T_{a_i}\rho_i - \rho_i I$$

is smoothing. Hence

$$P\gamma_i T_{a_i}\rho_i - \rho_i I \tag{4.7.2}$$

is smoothing as an operator on T^n. However, $P\gamma_i T_{a_i}\rho_i$ and $\rho_i I$ are *globally defined* as operators on X and hence (4.7.2) is a globally defined smoothing operator. Now let $Q = \sum \gamma_i T_{a_i}\rho_i$ and note that by (4.7.2)

$$PQ - I$$

is a smoothing operator.

□

This concludes the proof of Theorem 4.3, and hence, modulo proving Theorem 4.3. This concludes the proof of our main result: Theorem 4.2. The proof of Theorem 4.3 will be outlined, as a series of exercises, in the next section.

4.8 The Fredholm theorem for smoothing operators

Let X be a compact n-dimensional manifold equipped with a smooth non-vanishing measure, dx. Given $K \in \mathcal{C}^\infty(X \times X)$ let

$$T_K : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

be the smoothing operator 3.1.

Exercise 1. Let V be the volume of X (i.e., the integral of the constant function, 1, over X). Show that if

$$\max |K(x,y)| < \frac{\epsilon}{V}\,, \quad 0 < \epsilon < 1$$

then $I - T_K$ is invertible and its inverse is of the form, $I - T_L$, $L \in \mathcal{C}^\infty(X \times X)$.
Hint 1. Let $K_i = K \circ \cdots \circ K$ (i products). Show that $\sup |K_i(x,y)| < C\epsilon^i$ and conclude that the series

$$\sum K_i(x,y) \tag{4.8.1}$$

converges uniformly.
Hint 2. Let U and V be coordinate patches on X. Show that on $U \times V$

$$D_x^\alpha D_y^\beta K_i(x,y) = K^\alpha \circ K_{i-2} \circ K^\beta(x,y)$$

where $K^\alpha(x,z) = D_x^\alpha K(x,z)$ and $K^\beta(z,y) = D_y^\beta K(z,y)$. Conclude that not only does (8.1) converge on $U \times V$ but so do its partial derivatives of *all* orders with respect to x and y.

Exercise 2. (finite rank operators.) T_K is a finite rank smoothing operator if K is of the form:

$$K(x,y) = \sum_{i=1}^N f_i(x) g_i(y)\,. \tag{4.8.2}$$

(a) Show that if T_K is a finite rank smoothing operator and T_L is any smoothing operator, $T_K T_L$ and $T_L T_K$ are finite rank smoothing operators.

(b) Show that if T_K is a finite rank smoothing operator, the operator, $I - T_K$, has finite dimensional kernel and co-kernel.

Hint. Show that if f is in the kernel of this operator, it is in the linear span of the f_i's and that f is in the image of this operator if

$$\int f(y) g_i(y)\, dy = 0\,, \quad i = 1, \ldots, N\,.$$

Exercise 3. Show that for every $K \in \mathcal{C}^\infty(X \times X)$ and every $\epsilon > 0$ there exists a function, $K_1 \in \mathcal{C}^\infty(X \times X)$ of the form (4.8.2) such that

$$\sup |K - K_1|(x, y) < \epsilon\,.$$

Hint. Let $\mathcal{A}$ be the set of all functions of the form (4.8.2). Show that $\mathcal{A}$ is a *subalgebra* of $C(X \times X)$ and that this subalgebra separates points. Now apply the Stone–Weierstrass theorem to conclude that $\mathcal{A}$ is dense in $C(X \times X)$.

Exercise 4. Prove that if T_K is a smoothing operator the operator

$$I - T_K : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

has finite dimensional kernel and co-kernel.

Hint. Show that $K = K_1 + K_2$ where K_1 is of the form (4.8.2) and K_2 satisfies the hypotheses of exercise 1. Let $I - T_L$ be the inverse of $I - T_{K_2}$. Show that the operators

$$(I - T_K) \circ (I - T_L)$$
$$(I - T_L) \circ (I - T_K)$$

are both of the form: identity minus a finite rank smoothing operator. Conclude that $I - T_K$ has finite dimensional kernel and co-kernel.

Exercise 5. Prove Theorem 4.3.