Chapter 5

Hodge Theory

Lecture 19

(First see notes on Elliptic operators)

Let X be a compact manifold. We will show that Section 7 of the notes on Elliptic operators works for elliptic operators on vector bundles.

We'll be working with the basic vector bundles $TX \otimes \mathbb{C}$, $T^*X \otimes \mathbb{C}$, $\Lambda^1(T^*X) \otimes \mathbb{C}$ etc.

Let review the basic facts about vector bundle theory. $E \to X$ is a rank k (complex) vector bundle then given U open in X we define $E_U = E \mid_U$. Given $p \in U$ there exists an open set $U \ni p$ and a vector bundle isomorphism such that

$$\begin{array}{ccc} E & \xrightarrow{\;\cong\;} & U \times \mathbb{C}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & U & \end{array}$$

Notation. $C^\infty(E)$ denotes the C^∞ sections of E.

Suppose we have $E^i \to X$, $i = 1, 2$ vector bundles of rank k_i and suppose we have an operator $P : C^\infty(E^1) \to C^\infty(E^2)$.

Definition. P is an m**th order differential operator** if

(a) P is local. That is for every open set $U \subseteq X$ there exists a linear operator $P_U : C^\infty(E^1_U) \to C^\infty(E^2_U)$ such that $i^*_U P = P_U i^*_U$.

(b) If γ^i_U, $i = 1, 2$ are local trivializations of the vector bundle E^i over U then the operator $P^\sharp_U$ in the diagram below is an mth order differential operator

$$\begin{array}{ccc} C^\infty(E^1_U) & \xrightarrow{P_U} & C^\infty(E^2_U) \\ {\scriptstyle\gamma^1_U}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\gamma^2_U} \\ C^\infty(U, \mathbb{C}^{k_1}) & \xrightarrow{P^\sharp_U} & C^\infty(U, \mathbb{C}^{k_2}) \end{array}$$

Check: This is independent of choices of trivializations.

Let $p \in U$. From γ^i_U, $i = 1, 2$ we get a diagram (with $\xi \in T^*_p$)

$$\begin{array}{ccc} E^1_p & \xrightarrow{\sigma_\xi} & E^2_p \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow \\ \mathbb{C}^{k_1} & \xrightarrow{\sigma^\sharp_\xi} & \mathbb{C}^{k_2} \end{array} \qquad \sigma^\sharp_\xi = \sigma(P^\sharp_U)(p, \xi)$$

Definition. $\sigma_\xi = \sigma(P)(p, \xi)$

Check that this is independent of trivialization.
$f \in C^\infty(U)$, $s \in C^\infty(E_U)$. Then

$$(e^{-itf} P e^{itf})(p) = t^m \sigma(P)(p, \xi) s(p) + O(t^{m-1})$$

where $\xi = df_p$.

Definition. P is **elliptic** if $k_1 = k_2$ and for every p and $\xi \neq 0$ in T_pX, then $\sigma(P)(p, \xi) : E_p^1 \to E_p^2$ is bijective.

5.0.1 Smoothing Operators on Vector Bundles

We have bundles $E^i \to X$. Form a bundle $\mathrm{Hom}(E^1, E^2) \to X \times X$ by defining that at (x, y) the fiber of this bundle is $\mathrm{Hom}(E_x^1, E_y^2)$. In addition lets let dx be the volume form on X.

Let $K \in C^\infty(\mathrm{Hom}(E^1, E^2))$ and define $T_K : C^\infty(E^1) \to C^\infty(E^2)$, with $f \in C^\infty(E^1)$ by

$$T_K f(y) = \int K(x, y) f(x) dx$$

What does this mean? By definition $f(x) \in E_x^1$ and $K(x, y) : E_x^1 \to E_y^2$, so $(K(x, y)f(x)) \in E_y^2$. Thus it makes perfect sense to do the integration in the definition.

Theorem. *$P : C^\infty(E^1) \to C^\infty(E^2)$is an mth order elliptic differential operator, then there exists an "mth order ΨDO", $Q : C^\infty(E^2) \to C^\infty(E^1)$ such that*

$$PQ - I$$

is smoothing.

Proof. Just as proof outlined in notes with U_i, ρ_i, γ_i. But make sure that E^1, E^2 are locally trivial over U_i, i.e. on U_i, $P_{U_i} \cong P_{U_i}^\sharp$, so $P_{U_i}^\sharp$ is an elliptic system. □

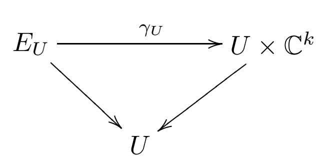
5.0.2 Fredholm Theory in the Vector Bundle Setting

Let $E \to X$ be a complex vector bundle. Then a hermitian inner product on E is a smooth function $X \ni p \to (,)_p$ where $(,)_p$ is a Hermitian inner product on E_p.

If X is compact with $s_1, s_2 \in C^\infty(E)$ then we can make this into a compact pre-Hilbert space by defining an L^2 inner product

$$\langle s_1, s_2 \rangle = \int (s_1(x), s_2(x)) dx$$

Lemma. *Given $p \in X$, there exists a neighborhood U of p and a Hermitian trivialization of E_U*



for $p \in U$, $E_p \cong \mathbb{C}^k$ and γ_U hermitian if $E_p \cong \mathbb{C}^k$ is an isomorphism of hermitian vector spaces.

Proof. This is just Graham-Schmidt □

Theorem. *$E^i \to X$, $i = 1, 2$ Hermitian vector bundles and $P : C^\infty(E^1) \to C^\infty(E^2)$ an mth order DO, then there exists a unique mth order DO, $P^t : C^\infty(E^2) \to C^\infty(E^1)$ such that for $f \in C^\infty(E^1)$, $g \in C^\infty(E^2)$*

$$\langle Pf, g \rangle_{L^2} = \langle f, P^t g \rangle_{L^2}$$

Proof. (Using the usual mantra: local existence, local uniqueness implies global existence global uniqueness). So we'll first prove local existence. Let U be open and γ_U^1, γ_U^2 hermitian trivialization of E_U^1, E_U^2. $P \leftrightsquigarrow P_U^\sharp$, $P_U^\sharp : C^\infty(U, \mathbb{C}^{k_1}) \to C^\infty(U, \mathbb{C}^{k_2})$. Then $P_U^\sharp = [P_{ij}]$, $P_{ij} : C^\infty(U) \to C^\infty(U)$, $1 \le i \le k_2$, $1 \le j \le k_1$.

Set $(P_U^t)^\sharp = [P_{ji}^t]$, $(P_U^t)^\sharp \rightsquigarrow P_U^t$. Then $P_U^t : C^\infty(E_U^2) \to C^\infty(E_U^1)$.

We leave the read to check that if $f \in C_0^\infty(E_U^1)$, $g \in C_0^\infty(E_U^2)$ then

$$\langle P_U f, g \rangle = \langle f, P_U^t g \rangle$$

This is local existence. Local uniqueness is trivial. This all implies global existence. □

Theorem (Main Theorem). *X compact, $E^i \to X$, $i = 1, 2$ hermitian bundles of rank k. And $P : C^\infty(E^1) \to C^\infty(E^2)$ an m order elliptic DO then*

(a) $\ker P$ *is finite dimensional*

(b) $f \in \operatorname{Im} P$ *if and only if* $\langle f, g \rangle = 0$ *for all* $g \in \ker P^t$.

Proof. The proof is implied by existence of right inverses for P modulo smoothing and the Fredholm Theorem for $I - T$ when $T : C^\infty(E^1) \to C^\infty(E^2)$. □