Lecture 20

X a compact manifold, $E^k \to X$, $k = 1, \ldots, N$ complex vector bundles, $D : C^\infty(E^k) \to C^\infty(E^{k+1})$ first order differential operator. Consider the following complex, hereafter referred to as $(*)$.

$$\cdots \longrightarrow C^\infty(E^k) \xrightarrow{D} C^\infty(E^{k+1}) \xrightarrow{D} \cdots$$

$(*)$ is a differential complex if $D^2 = DD = 0$.

For $x \in X$, $\xi \in T_x^*$, we have $\sigma_\xi : E_x^k \to E_x^{k+1}$ then we have the symbol $\sigma_\xi(D)(x, \xi)$. And

$$0 = \sigma(D^2)(x, \xi) = \sigma(D)(x, \xi)\sigma(D)(x, \xi)$$

so we conclude that $\sigma_\xi^2 = 0$. So at every point we get a finite dimensional complex

$$0 \longrightarrow E_x^1 \xrightarrow{\sigma_\xi} E_x^2 \xrightarrow{\sigma_\xi} \cdots$$

the **symbol complex**

Definition. $(*)$ is elliptic if the symbol complex is exact for all x and $\xi \in T_x^* - \{0\}$.

Examples

(a) The De Rham complex. For this complex the bundle is

$$E^k : \Lambda^k \otimes \mathbb{C} = \Lambda^k(T^*X) \otimes \mathbb{C}$$

then $C^\infty(E^k) = \Omega^k(X)$. The first order operation is the usual exterior derivative $d : C^\infty(E^k) \to C^\infty(E^{k+1})$. $\sigma_\xi = \sigma(d)(x, \xi)$, where $\sigma_\xi : \Lambda^k(T_x^*) \otimes \mathbb{C} \to \Lambda^{k+1}(T_x^*) \otimes \mathbb{C}$

Theorem. *For $\mu \in \Lambda^k(T_x^*) \otimes \mathbb{C}$, $\sigma_\xi \mu = \sqrt{-1}\xi \wedge \mu$.*

Proof. $\omega \in \Omega^k(X)$, $\omega_x = \mu$, $f \in C^\infty(X)$, $df_x = \xi$ then

$$(e^{-itf} de^{ift}\omega)_x = (idf \wedge \omega)_x + (d\omega)_x = (i\xi_x \wedge \mu)t + (d\omega)_x$$

Theorem. *The de Rham complex is elliptic*

Proof. To do this we have to prove the exactness of the symbol complex:

$$\cdots \longrightarrow \Lambda^k(T_x^*) \xrightarrow{\text{“}\wedge\xi\text{”}} \Lambda^{k+1}(T_x^*) \xrightarrow{\text{“}\wedge\xi\text{”}} \cdots$$

To do this let $e_1, \ldots, e_n$ be a basis of T_x^* with $e_1 = \xi$. Then for $\mu \in \Lambda^k(T_x^*)$, $\mu = e_1 \wedge \alpha + \beta$ where α and β are products just involving $e_2, \ldots, e_n$ (this is not hard to prove). $\square$

(b) Let X be complex and let us define a vector bundle

$$E^k = \Lambda^{0,k}(T^*) \qquad C^\infty(E^k) = \Omega^{0,k}(X)$$

Take $D = \overline{\partial}$. This is a first order DO,

$\overline{\partial} : C^\infty(E^k) \to C^\infty(E^{k+1})$, $\sigma_x i = \sigma(D)(x,\xi)$, now what is this symbol?

Take $\xi \in T^*_x$, then $\xi = \xi^{1,0} + \xi^{0,1}$ where $\xi^{1,0} \in (T^a st_x)^{1,0}, \xi^{0,1} \in (T^*_x)^{0,1}$ and $\xi^{1,0} = \overline{\xi}^{0,1}$, $\xi \neq 0$ then $\xi^{0,1} \neq 0$.

Theorem. *For $\mu \in \Lambda^{0,ki}(T^*_x)$, $\sigma_\xi(\mu) = \sqrt{-1}\xi^{0,1} \wedge \mu$.*

Proof. $\omega \in \Omega^{0,k}(X)$, $\omega_x = \mu$, $f \in C^\infty(X)$, $df_x = \xi$ then

$$(e^{-itf}\overline{\partial}e^{itf}\omega)_x = (it\overline{\partial}f \wedge \omega)_x t + (\overline{\partial}\omega)_x = it\xi^{0,1} \wedge \mu + \overline{\partial}\omega_x$$

□

Check: For $\xi \neq 0$ the sequence

$$\cdots \longrightarrow \Lambda^{0,k}(T^*_x) \xrightarrow{\text{“}\wedge\xi^{0,1}\text{”}} \Lambda^{0,k+1}(T^*_x) \xrightarrow{\text{“}\wedge\xi^{0,1}\text{”}} \cdots$$

is exact. This is basically the same as the earlier proof, when we note that $\Lambda^{0,k}(T^*_x) = \Lambda^k((T^*_x)^{0,1})$. we conclude that the Dolbeault complex is elliptic.

(c) The above argument forks for higher dimensional Dolbeault complexes. If we set

$$E^k = \Lambda^{p,k}(T^*X), \qquad D = \overline{\partial}, \qquad C^\infty(E^k) = \Omega^{p,k}(X)$$

it is easy to show that $\sigma(\overline{\partial})(x,\xi) =$ “$\wedge\, \xi^{0,1}$”

The Hodge Theorem

Given a general elliptic complex

$$\cdots \xrightarrow{D} C^\infty(E^k) \xrightarrow{D} C^\infty(E^{k+1}) \xrightarrow{D} \cdots$$

with dx a volume form on X, equip each vector bundle E^k with a Hermitian structure. We then get an L^2 inner product $\langle,\rangle_{L^2}$ on $C^\infty(E^k)$. And for each $D : C^\infty(E^k) \to C^\infty(E^{k+1})$ we get a transpose operator

$$D^t : C^\infty(E^{k+1}) \to C^\infty(E^k)$$

If for $x \in X$, $\xi \in T^*_x$, $\sigma_\xi = \sigma(D)(x,\xi)$ then

$$\sigma(D^t)(x,\xi) = \sigma^t_x$$

So we can get a complex in the other direction, call it $(*)^t$

$$\cdots \xrightarrow{D^t} C^\infty(E^k) \xrightarrow{D^t} C^\infty(E^{k-1}) \xrightarrow{D^t} \cdots$$

and since $0 = (D^r)^t = (DD)^t = D^tD^t = (D^t)^2$ we have that $(*)^t$ is a differential complex.

Also, $\sigma(D^t)(x,\xi) = \sigma_\xi = \sigma(D)(x,\xi)^t$. For x and $\xi \in T_x^* - \{0\}$ the symbol complex of D^t is

$$0 \longrightarrow E_x^N \xrightarrow{\sigma_\xi^t} E_x^{N-1} \xrightarrow{\sigma_\xi^t} \cdots$$

The transpose of the symbol complex for D. So $(*)$ elliptic implies that $(*)^t$ is elliptic.

Definition. The **harmonic space** for $(*)$ is

$$\mathcal{H}^k = \{s \in C^\infty(E^k), Ds = D^ts = 0\}$$

Theorem (Hodge Decomposition Theorem). *We have two propositions*

(a) For all k, $\mathcal{H}^k$ is finite dimensional.

(b) Every element u of $C^\infty(E^k)$ can be written uniquely as a sum $u_1 + u_2 + u_3$ where $u_1 \in \mathrm{Im}(D)$, $u_2 \in \mathrm{Im}(D^t)$, $u_3 \in \mathcal{H}^k$

Before we prove this we'll do a little preliminary work. Let

$$E = \bigoplus_{k=1}^{N} E^k$$

Then consider the operator

$$D + D^t : C^\infty(E) \to C^\infty(E)$$

Check: This is elliptic.

Proof. Consider $Q = (D + D^t)^2$. It suffices to show that Q is elliptic.

$$Q = D^2 + DD^t + D^tD + (D^t)^2$$

but the two end terms are 0. So

$$Q = DD^t + D^tD$$

Note that Q sends $C^\infty(E^k)$ to $C^\infty(E^k)$, so Q behaves nicer than $D + D^t$. So now we want to show that Q is elliptic.

Let x, $\xi \in T_x^* - \{0\}$. Then

$$\sigma(Q)(x,\xi) = \sigma(DD^t)(x,\xi) + \sigma(D^tD)(x,\xi) = \sigma_x^t\xi_\xi + \sigma_\xi\sigma_\xi^t$$

(where $\sigma_\xi = \sigma(D)(x,\xi)$.

Suppose $v \in E_x^k$ and $\sigma(Q)(x,\xi)v = 0$ (i.e. it fails to be bijective). Then

$$((\sigma_\xi^t\sigma_\xi + \sigma_\xi\sigma_\xi^t)v, v) = 0 = (\sigma_\xi v, \sigma_\xi v)_x + (\sigma_\xi^t v, \sigma_\xi^t v) = 0$$

which implies that $\sigma_\xi v = 0$ and $\sigma_\xi^t v = 0$. Now $\sigma_\xi = 0$ implies that $v \in \mathrm{Im}\,\sigma_\xi : E_x^{k-1} \to E_x^k$ by exactness. We know that $\mathrm{Im}\,\sigma_\xi \perp \ker \sigma_\xi^t$, but $v \in \ker \sigma_\xi^t$, so $v \perp v$ implies that $v = 0$.

So Q is elliptic and thus $(D + D^t)$ is elliptic. ∎

Lemma. $\mathcal{H}^k = \ker Q$.

Proof. We want to show $\mathcal{H}^k \subseteq \ker Q$. The other direction is easy. Let $u \in \ker Q$. Then

$$\langle DD^t u + D^t Du, u\rangle = 0 = \langle D^t u, D^t u\rangle + \langle Du, Du\rangle = 0$$

This implies that $D^t u = Du = 0$, so $u \in \mathcal{H}^k$. □

Proof of Hodge Decomposition. By the Fredholm theorem every element $u \in C^\infty(E^k)$ is of the form $u = v_1 + v_2$ where $v_1 \in \mathrm{Im}(Q)$ and $v_2 \in \ker Q$. $v_2 \in \ker Q$ implies that $v_2 \in \mathcal{H}^k$, $v_1 \in \mathrm{Im}\, Q$ implies that $v_1 = Qw = D(D^t w) + D^t(Dw)$. Choose $u_1 = DD^t w, u_2 = D^t Dw$ and $v_2 = u_3$. □

Left as an exercise: Check that $u = u_1 + u_2 + u_3$ is unique. Hint: $\ker D \perp \mathrm{Im}\, D^t$ and $\ker D^t \perp \mathrm{Im}\, D$. Then the space $\mathrm{Im}(D)$, $\mathrm{Im}(D^t)$ and $\mathcal{H}$ are all mutually perpendicular.