Lecture 22

Again, $V = V^n$ and $B : V \times V \to \mathbb{R}$ a non-degenerate bilinear form. A few properties of $*$ we have not mentioned yet:

$$*1 = \Omega \qquad * \Omega = 1$$

Computing the $*$-operator

We now present a couple of applications to computation

(a) B symmetric and positive definite. Let $v_1, \dots, v_n$ be an oriented orthonormal basis of V. If $I = (i_1, \dots, i_k)$ where $i_1 < \cdots < i_k$ then $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$. Let $J = I^C$. Then

$$*v_I = \pm v_J$$

where this is postive if $v_I \wedge v_J = \Omega$ and negative if $v_I \wedge v_J = -\Omega$.

(b) Let B be symplectic and $V = V^{2n}$. Then there is a Darboux basis $e_1, f_1, \dots, e_n, f_n$. Give V the symplectic orientation

$$\Omega = e_1 \wedge f_1 \wedge \cdots \wedge e_n f_n$$

What does the $*$-operator look like? For $n = 1$, i.e. $V = V^2$ we have $*1 = e \wedge f$,$*(e \wedge f) = 1$ $*e = e$ and $*f = f$.

What about n arbitrary? Suppose we have

$$V = V_1 \oplus \cdots \oplus V_n \qquad V_i = span\{e_i, f_i\}$$

then $\Lambda(V)$ is spanned by $\beta_1 \wedge \cdots \wedge \beta_n$ where $\beta_i \in \Lambda^{p_i}(V_i)$, $0 \le p_i \le 2$. Then

$$*(\beta_1 \wedge \cdots \wedge \beta_n) = *_n\beta_n \wedge \cdots \wedge *_1\beta_1$$

and we already know that $*$ operator on 2 dimensional space.

Other Operations

For $u \in V$ we can define an operation $L_u : \Lambda^k \to \Lambda^{k+1}$ by $\alpha \mapsto u \wedge \alpha$. We can also define this operations dual: for $v^* \in V^*$, $i_{v^*} : \Lambda^k \to \Lambda^{k-1}$ the usual interior product.

But because we have a bilinear form we can find L_u^t and $i_{v^*}^t$ and since we have $*$ we have other interesting things to do, like conjugate with the $*$-operator:

$$*^{-1} L_u * \qquad *^{-1}(i_{v^*})*$$

Theorem. *For $\alpha \in \Lambda^{p-1}$, $\beta \in \Lambda^p$*

$$B(L_u\alpha, \beta) = B(\alpha, L_u^t\beta)$$

where $L_u^t = (-1)^{p-1} *^{-1} L_u * := \widetilde{L}_u$.

Proof. Begin by noting $L_u\alpha \wedge *\beta = B(L_u\alpha, \beta)\Omega$. Now

$$\begin{aligned} u \wedge \alpha \wedge *\beta = (-1)^{p-1}\alpha \wedge u \wedge *\beta &= (-1)^p \alpha \wedge *(*^{-1}u \wedge *\beta) \\ &= \alpha \wedge *\widetilde{L}_u\beta = B(\alpha, \widetilde{L}_u\beta)\Omega \end{aligned}$$

which implies that $\widetilde{L}_u = L_u^t$. □

What is this transpose really doing? We know we have a bilinear form B that gives rise to an map $L_u : V \to V^*$. Since B is not symmetric, define $B^\sharp(u, v) = B(v, u)$, and we get a new map $L_{B^\sharp} : V \to V^*$. Then:

Theorem. *If $v^* = L_{B^\sharp} u$, then $L_u^t = i_{v^*}$.*

Proof. Let $u_1, \ldots, u_n$ be a basis of V and let $v_1, \ldots, v_n$ be a complementary basis of V determined by

$$B(u_i, v_j) = \delta_{ij}$$

and let $v_1^*, \ldots, v_n^*$ be a dual basis of V^*. Check that $v_1^* = L_{B^\sharp} u_1$. Let $I = (i_1, \ldots, i_{k-1})$ and $J = (j_1, \ldots, j_k)$ be multi-indices. We claim that

$$B(L_{u_1} u_I, v_J) = B(u_I, i_{v_1^*} v_J)$$

and that if $j_1, \ldots, j_k = 1$ and $i_1, \ldots, i_{k-1} = 1$ then both sides are 1. Otherwise they are 0.

Theorem. *On Λ^{p+1}, $(i_{v^*})^t = (-1)^p *^{-1} (i_{v^*}) *$ and $v^* = L_B u$.*