Lecture 26

Lemma. *Take $v \in V$, $Hv = \lambda v$. We claim that $H(Xv) = (\lambda + 2)Xv$.*

Proof. $(HX - XH)v = 2Xv$, so $HXv = \lambda Xv + 2Xv = (\lambda + 2)Xv$.

Lemma. *If $Hv = \lambda v$, then*

$$[X, Y^k]v = k(\lambda - (k-1))Y^{k-1}v$$

□

Proof. We proceed by induction. If $k = 1$ this is just $[X, Y]v = Hv = \lambda v$. This is true.
Now we show that if this is true for k, its true for $k + 1$.

$$\begin{aligned}
[X, Y^{k+1}]v &= XY^{k+1}v - Y^{k+1}Xv \\
&= (XY)Y^k v - (YX)Y^k v + Y(XY^k)v - Y(Y^k Xv) \\
&= HY^k v + Y([X, Y^k])v \\
&= (\lambda - 2k)Y^k v + Y(k(\lambda - (k-1))Y^{k-1}v \\
&= ((\lambda - 2k) + k(\lambda - k - 1))Y^k v = (k+1)(\lambda - k)Y^k v
\end{aligned}$$

Definition. V is a cyclic module with generator v if every submodule of V containing v is equal to V itself.

Theorem. *If V is a cyclic module of finite H type then* $\dim V < \infty$.

□

Proof. Let v generate V. Then $v = \sum_{i=0}^{N} v_i$ where $v_i \in V_i$. It is enough to prove the theorem for cyclic modules generated by v_i. We can assume without loss of generality that $Hv = \lambda v$.

Now, note that only a finite number of expression $Y^k X^l v$ are non-zero (since X shifts into a different eigenspace, and there are only a finite number of eigenspaces).

By the formula that we just proved, *span*$\{Y^k X^l v\}$ is a submodule of V containing v.

□

Fact: Every finite dimensional $\mathbf{g}$-module is a direct sum of irreducibles.
In particular, every cyclic submodule of V is a direct sum of irreducibles.

Theorem. *Every irreducible* $\mathbf{g}$*-module of finite H type is of the form $V = V_0 \oplus \cdots \oplus V_k$ where* $\dim V_i = 1$. *Moreover, there exists $v_i \in V_i - \{0\}$ such that*

$$\begin{aligned}
&Hv_i = (k - 2i)v_i \\
&Yv_i = v_{i+1} \qquad i \leq k-1 \\
&Xv_i = i(k - (i-1))v_{i-1} \qquad i \geq 1 \\
&Xv_0 = 0, Yv_k = 0
\end{aligned}$$

Proof. Let $V = V_0 \oplus \cdots \oplus V_n$, and $H = \lambda_i Id$ on V_i and assume that $\lambda_0 > \lambda_1 > \cdots > \lambda_n$. Take $v \in V_0 - \{0\}$. Note that $Xv = 0$, because $HXv = (\lambda_0 + 2)Xv$ and $\lambda_0 + 2 > \lambda_0$.

Consider $Yv, \ldots, Y^k v \neq 0$, $Y^{k+1}v = 0$, so $HY^i v = (\lambda_0 - 2i)Y^i v$. and

$$XY^i v = Y^i Xv + i(\lambda - (i-1))Y^{i-1}v = i(\lambda - (i-1))Y^{i-1}v$$

When $i = k + 1$ we have

$$XY^{k+1}v = 0 = (k+1)(\lambda - k)Y^k v$$

but $Y^k v \neq 0$, so it must be that $\lambda = k$. Now just set $v_i = Y^i v$.

□

Lemma. *Let V be a $k+1$ dimensional vector space with basis $v_0, \dots, v_k$. Then the relations in the above theorem define an irreducible representation of* $\mathbf{g}$ *on V*

Definition. V a $\mathbf{g}$-module, $V = \bigoplus_{i=0}^{N} V_i$ of finite H-type. Then $v \in V$ is **primitive** if

(a) v is homogenous,(i.e. $v \in V_i$)

(b) $Xv = 0$.

Theorem. *If v is primitive then the cyclic submodule generated by v is irreducible and $Hv = k$ where k is the dimension of this module.*

Proof. $v, Yv, \dots, Y^k v \neq 0$, $Y^{k+1} = 0$. Take $v_i = Y^i v$. Check that v_i satisfies the conditions. □

Theorem. *Every vector $v \in V$ can be written as a finite sum*

$$v = \sum Y^l v_l$$

where v_l is primitive.

Proof. This is clearly true if V is irreducible (by the relations). Hence this is true for cyclic modules, because they are direct sums of irreducibles, hence this is true in general. □

Corollary. *The eigenvalues of H are integers.*

Proof. We need to check this for eigenvectors of the form $Y^l v$ where v is primitive. But for v primitive we know the theorem is true, i.e. $Hv = kv$, $HY^l v = (k-2l)Y^l v$. So write $V = \bigoplus V_r$, $H = rId$ on V_r □