Lecture 27

Theorem. *We can repagenate the sum so that*

$$V = \bigoplus_{i=-N}^{N} V_i$$

where

$H = iId$ *on* V_i

(a) $X : V_i \to V_{i+2}$ *and* $Y : V_{i+2} \to V_i$.

(b) $Y^i V_i \xrightarrow{cong} V_{-i}$ *is bijective.*

Now, recall that we are going to apply this stuff to Hodge Theory. In particular, let (X^{2n}, ω) be a symplectic, compact manifold. Then we define $L : \Omega^k(X) \to \Omega^{k+2}(X)$ given by $\alpha \mapsto \omega \wedge \alpha$, $* : \Omega^k \to \Omega^{2n-k}$, $L^t : \Omega^{k+2} \to \Omega^k$ given by $L^t = *L*$ and we defined $A : \Omega \to \omega$, $A = iId$ on Ω^{n-i}. The Kaehler-Weil identities said that

$$[L^t, L] = A \qquad [A, L^t] = 2L^t \qquad [A, L] = -2L$$

So Ω is a $\mathbf{g}$-module of finite H-type with $X = L^t$, $Y = L$ and $H = A$.

Corollary. *The map* $L^k : \Omega^{n-k} \to \Omega^{n+k}$ *is an isomorphism.*

We can apply this to symplectic hodge theory as follows. We know in this case that

$$[d, L^t] = \delta \qquad [\delta, L] = d$$

Let $\Omega_{harm} = \{u \in \Omega du = \delta = 0\}$.

Theorem. Ω_{harm} *is a* $\mathbf{g}$*-module of* Ω.

Corollary. *The map* $L^k : \Omega^{n-k}_{harm} \to \Omega^{n+k}_{harm}$ *is bijective.*

Hard Lefshetz Theorem

$\omega \in \Omega^2$, $d\omega = 0$. Then $[\omega]$ defines a cohomology class $[\omega] \in H^2_{DR}(X) = H^2(X)$. And in turn we can define a mapping $\gamma : H^k(X) \to H^{k+2}(X)$ by $c \mapsto [\omega] \frown c$.

Theorem. *Let* X *be Kaehler then* $\gamma^k : H^{n-k}(X) \to H^{n+k}(X)$ *is bijective.*

What about the symplectic case? Let $u \in \Omega^k_{harm}$ with $du = 0$. Define a mapping $P_k : \Omega^k_{harm} \to H^k(X)$ by $u \mapsto [u]$

Theorem. *(Matthieu) Hard Lefshetz holds for* X *if and only if* P_x *is onto for all* k.

Proof. The "only if" part is covered in the supplementary notes. Now the for the "if" part, we use the following diagram

$$\begin{array}{ccc} \Omega^{n-k}_{harm} & \xrightarrow{L^k} & \Omega^{n+k}_{harm} \\ \downarrow & & \downarrow \\ H^{n-k}(X) & \xrightarrow{\gamma^k} & H^{n+k}(X) \end{array}$$

L^k is bijective, the vertical arrows are surjective, so γ^k is surjective. Poincare duality tells us that $\dim H^{n-k} = \dim H^{n+k}$ so γ^k is bijective. ∎

Remarks:

(a) "if" condition is automatic for Kaehler manifolds

(b) A consequence of Hard Lefshetz. We know that $H^{2n}(X) \xrightarrow{\cong} \mathbb{R}$ given by $[u] \mapsto \int_X u$ is (by stokes theorem) bijective. Hence one can define a bilinear form on $H^{n-k}(X)$ via

$$c_1, c_2 \to \gamma^k c_1 \frown c_2 \in H^{2n}(X) \xrightarrow{\cong} \mathbb{R}$$

By poincare and hard lefshetz this form is non-degenerate, i.e. $\gamma^k c_1 \frown c_2 = 0$ for all c_2, then by Poincare $\gamma^k c_1 = 0$ which implies that $c_1 = 0$.

A consequence is that for k odd $H^k(X)$ is even dimensional.

(c) Thurston showed that there exists lots of compact symplectic manifolds with $\dim H^1(X)$ odd, i.e. it doesn't satisfy strong lefshetz.

(d) For any symplectic manifold X, let $H^k_{symp}(X) = \operatorname{Im}(\Omega^k_{harm} \to H^k(X))$. For symplectic cohomology you **do** have Hard Lefshetz.

Riemannian Hodge Theory

Let $V = V^n$ be a vector space over $\mathbb{R}$. B is a positive definite inner product on V. Assume V is oriented, then you get $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$. Take $v_1, \dots, v_n$ to be an oriented orthonormal basis of V. $I = (i_1, \dots, i_k)$, $i_1 < \cdots < i_k$. I^c the complementary multi-index. Then $*v_I = \epsilon v_{I^c}$ where $\epsilon v_I \wedge v_{I^c} = v_1 \wedge \cdots \wedge v_n$ (where ϵ is some sign).

Let $X = X^n$ be a compact Riemannian manifold. From the Riemannian metric we get B_p a positive definite inner product on T^*_p so B_p induces a positive definite inner product on $\Lambda^k(T^*_p)$.

From these inner products we get the star operator $*_p : \Lambda^k_p \to \Lambda^{n-k}_p$ satisfying $\alpha, \beta \in \Lambda^k_p$, $\alpha \wedge *\beta = B_p(\alpha, \beta) v_p$ where v_p is the Riemannian volume form.

Its clear that B_p extends $\mathbb{C}$-linearly to a $\mathbb{C}$-blinear form on $\Lambda^k_p \otimes \mathbb{C}$ and $*_p$ extends $\mathbb{C}$-linearly to $\Lambda^k_p \otimes \mathbb{C}$.

A hermitian inner product on $\Lambda^k(T^*_p) \otimes \mathbb{C}$ by $(\alpha, \beta)_p = B_p(\alpha\ \bar{\beta})$ and $\alpha \wedge *\bar{\beta} := (\alpha, \beta)_p v_p$.

Globally, $\Omega^k(X) = C^\infty(\Lambda^k(T^*X) \otimes \mathbb{C})$. Define an L^2 inner-product by $\alpha, \beta \in \Omega^k(X)$

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta)_p v = \int_X \alpha \wedge *\bar{\beta}$$

From $\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \dots$ we get an elliptic complex

$$C^\infty(X) \longrightarrow C^\infty(\Lambda^1(T^*X) \otimes \mathbb{C}) \longrightarrow \cdots$$

We have a hermitian inner product on the vector bundles $\Lambda^k(T^*X) \otimes \mathbb{C}$, so we can get a transpose

$$d^t : C^\infty(\Lambda^k(T^*X) \otimes \mathbb{C}) \to C^\infty(\Lambda^{k-1}(T^*X) \otimes \mathbb{C})$$

and write $d^t = \delta$ and think of δ as $\delta : \Omega^k \to \Omega^{k-1}$.

Form the corresponding Laplacian operator $\Delta = d\delta + \delta d$.

Apply the general theory of Elliptic complexes to this case. We conclude that

(a) $\mathcal{H}^k = \{u \in \Omega^k, \Delta u = 0\}$ is finite dimensional.

(b) $\mathcal{H}^k = \{u \in \Omega^k, du = \delta u = 0\}$.

(c) Hodge Decomposition

$$\Omega^k = \{(\operatorname{Im} d) \oplus (\operatorname{Im} \delta) \oplus \mathcal{H}^k\}$$

(d) The map $\mathcal{H}^k \to H^k_{DR}$ is bijective, i.e. every cohomology class has a unqiue harmonic representation.