Lecture 28

The H^k_{DR} are finite-dimensional.

Poincare Duality

Make a pairing $P : \Omega^k \times \Omega^{n-k} \to \mathbb{C}$ given by

$$P(\alpha, \beta) = \int_X \alpha \wedge \beta$$

If α is exact and β closed then $P(\alpha, \beta) = 0$, since $\alpha = d\omega$, $d\beta = 0$ and $\alpha \wedge \beta = du \wedge \beta = d(u \wedge \beta)$. By stokes $\int \alpha \wedge \beta$ is thus 0. P induces a pairing in cohomology, $P^\sharp : H^k_{DR} \times H^{n-k}_{DR} \to \mathbb{C}$.

Theorem (Poincare). *This is a non-degenerate pairing.*

We give a Hodge Theoretic Proof. First,

Lemma. *$\delta : \Omega^k \to \Omega^{k-1}$ is given by $\delta = (-1)^k *^{-1} d*$*

Proof. Let $\delta_1 = (-1)^k *^{-1} d*$, we want to show that $\delta = \delta_1$. Let $\alpha \in \Omega^{k-1}$ and $\beta \in \Omega^{n-k}$ then

$$\begin{aligned} d(\alpha \wedge \bar{\beta}) &= d\alpha \wedge \bar{\beta} + (-1)^{k-1} \alpha \wedge d * \bar{\beta} \\ &= d\alpha \wedge *\bar{\beta} + (-1)^{k-1} \alpha \wedge *(*^{-1} d * \bar{\beta}) \\ &= d\alpha \wedge *\bar{\beta} - \alpha \wedge *(\overline{\delta_1 \beta}) \end{aligned}$$

Now integrate and apply stokes

$$\int d\alpha \wedge *\bar{\beta} = \int \alpha \wedge *\delta_1 \beta$$

so $\langle d\alpha, \beta \rangle = \langle \alpha, \delta_1 \beta \rangle$ and $\delta_1 = d^t = \delta$. □

Corollary. $*\mathcal{H}^k = \mathcal{H}^{n-k}$

Proof. Take $\alpha \in \mathcal{H}^k$. We'll show that $d * \alpha = 0$. This happens iff $*^{-1} d * \alpha = \pm \delta \alpha$. Since $\delta \alpha = 0$, $d * \alpha = 0$. It is similarly easy to check that $\delta * \alpha = 0$. □

Proof of Poincare Duality. If suffices to check that the pairing $P : \mathcal{H}^k \times \mathcal{H}^{n-k} \to \mathbb{C}$ given by $\alpha, \beta \mapsto \int_X \alpha \wedge \beta$ is non-degenerate.

Suppose $P(\alpha, \beta) = 0$ for all β. Take $\beta = *\bar{\alpha}$. Then

$$P(\alpha, \beta) = \int_X \alpha \wedge *\bar{\alpha} = \langle \alpha, \alpha \rangle = 0$$

so this would imply that $\alpha = 0$. □

A Review of Kaehlerian Linear Algebra

Definition. $V = V^{2n}$ a vector space over $\mathbb{R}$, B_s a non-degenerate alternating bilinear form on V, $J : V \to V$ a linear map such that $J^2 = -I$. B_s and J are compatible if $B_s(Jv, Jw) = B_s(v, w)$.

Lemma. *If B_s and J are compatible if and only if the bilinear form $B_r(v, w) = B_s(v, Jw)$ is symmetric. (Here B_r is a Riemannian metric)*

J, B_s Kaehler implies that B_r is positive definite.

Notice that $B_r(Jv, Jw) = B_s(Jv, J^2 w) = B_s(v, Jw) = B_r(v, w)$ so that B_r and J are compatible. And also notice that $B_r(Jv, w) = B_s(Jv, Jw) = B_s(v, w)$. Let J^t be the transpose of J with respect to B_r Then

$$B_r(Jv, Jw) = B_r(v, J^t J w) = B_r(v, w)$$

so $J^t J = I$ and $J^t = -J$.

B_r, B_s, J in Coordinates

Let $e \in V$ such that $B_r(e,e) = 1$, and set $f = Je$, and $e = -Jf$. Then

$$B_r(e,e) = 1 \qquad B_s(e,f) = 1$$

Take $V_1 = span\{e,f\}$. This is a J-invariant subspace. If we then take

$$V_1^{\perp} = \text{orthocomplement of } V_1 \text{ w.r.t } B_r$$

then for $v \in V_1, w \in V_1^{\perp}$, $0 = B_r(Jv,w) = B_s(v,w)$, so $V_1^{\perp}$ is the symplectic orthocomplement of V_1 with respect to B_s.

Applying induction we get a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

where $V_i = span\{e_i, f_i\}$ such that $e_1, f_1, \ldots, e_n, f_n$ is an oriented orthonormal basis of V with respect to B_r and a Darboux basis with respect to B_s. Note that $Je_i = f_i$ and $Jf_i = -e_i$

5.0.3 B_r, B_s and J on $\Lambda^k(V)$

$\omega = \sum e_i \wedge f_i$ is the symplectic element in $\Lambda^2(V)$ and $\Omega = \omega^n/n! = e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n$ is the symplectic volume for and Riemannian volume form.

On decomposable elements, $\alpha = v_1 \wedge \cdots \wedge v_k$ and $\beta = w_1 \wedge \cdots \wedge w_k$ and

$$B_r(\alpha,\beta) = \det(B_r(v_i,w_j)) \qquad B_s(\alpha,\beta) = \det(B_s(v_i,w_j))$$

and we can define

$$J\alpha = Jv_1 \wedge \cdots \wedge Jv_k$$

Notice that

$$B_r(\alpha,\beta) = \det(B_r(v_i,w_j)) = \det B_s(v_i, Jw_j) = B_s(\alpha, J\beta)$$

and furthermore, it is easy to check that $B_r(J\alpha,J\beta) = B_r(\alpha,\beta)$, $B_s(J\alpha,J\beta) = B_s(\alpha,\beta)$, $J^2 = (-1)^k Id$ and if $J^t : \Lambda^k \to \Lambda^k$ is the B_r-transpose of J, then $J^t = (-1)^k J$.

The Star Operators

These are $*_r$ and $*_s$, the Riemannian and symplectic star operators, respectively. Let Ω be the symplectic (and Riemannian) volume form. For $\alpha, \beta \in \Lambda^k$ we have

$$\alpha \wedge *_r\beta = B_r(\alpha,\beta)\Omega = B_s(\alpha, J\beta) = \alpha \wedge *_s J\beta$$

so

$$*_r = *_s J$$

Also, notice that

$$J\alpha \wedge *_r J\beta = B_r(J\alpha, J\beta)\Omega = B_r(\alpha,\beta)\Omega = \alpha \wedge *_r\beta$$

on the other hand $J\Omega = \Omega$, so

$$\alpha \wedge *_r\beta = B_r(\alpha,\beta)\Omega = J\alpha \wedge *_r J *_r \beta$$

so $*_r J = J *_r$ and since $*_r = *_s J$ we have $J *_s = *_s J$.

Structure of $\Lambda(V)$

We have a symplectic element $\omega = \sum e_i \wedge f_i \in \Omega^2$. From this, we can define a mapping $L : \Lambda^k \to \Lambda^{k+2}$ given by $\alpha \mapsto \omega \wedge \alpha$. Note that

$$LJ\alpha = \omega \wedge J\alpha = J(\omega \wedge \alpha) = JL\alpha$$

so that $[J, L] = 0$.

Similarly for $L^t : \Lambda^{k+2} \to \Lambda^k$, the symplectic transpose given by $L^t = *_s L *_s$. Since $*_s, L$ commute with the J map, so does L^t, so $[J, L^t] = 0$.

Notice that

$$B_r(L\alpha,\beta) = B_s(L\alpha, J\beta) = B_s(\alpha, L^t J\beta) = B_s(\alpha, JL^t\beta) = B_r(\alpha, L^t\beta)$$

so L^t is also the Riemannian transpose.

From L, L^t we get a representation of $SL(2,\mathbb{R})$ on $\Lambda(V)$ and this representation is J-invariant.