Lecture 29

We now extend $*_r, *_s, J, L, L^t$, $\mathbb{C}$-linearly to $\Lambda^* \otimes \mathbb{C}$. And extend B_r, B_s to $\mathbb{C}$-linear forms on $\Lambda^k \otimes \mathbb{C}$.

We can now take $\Lambda^1 \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$, where as usual the two elements of the splitting are the eigenspaces of the J operator.

If we now let $e_1, f_1, \ldots, e_n, f_n$ be a Kaehlerian Darboux basis of V and set

$$u_i = \frac{1}{2\sqrt{-1}}(e_i - \sqrt{-1} f_i)$$

then $u_1, \ldots, u_n$ is an orthonormal basis of $\Lambda^{1,0}$ with respect to the Hermitian form $(u, v) = B_r(u, \bar{v})$ and $\bar{u}_1, \ldots, \bar{u}_n$ is an orthonormal basis of $\Lambda^{0,1}$.

We know from earlier that $*$ gives rise to a splitting

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}$$

and if I and J are multi-indices of length p and q, then the $u_I \wedge \bar{u}_J$ forms form an orthonormal basis of $\Lambda^{p,q}$ with respect to the Riemannian bilinear form $(\alpha, \beta) = B_r(\alpha, \bar{\beta})$.

In particular $\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q} \Lambda^{p+q}$ is an orthonormal decomposition of $\Lambda^k \otimes \mathbb{C}$ with respect to the inner product $(\alpha, \beta) = B_r(\alpha, \bar{\beta})$.

In terms of $u_1, \ldots, u_n \in \Lambda^{1,0}$, the symplectic form is

$$\omega = \frac{1}{2\sqrt{-1}} \sum u_i \wedge \bar{u}_i \in \Lambda^{1,1}$$

Consequences:

(a) $L : \Lambda^{p,q} \to \Lambda^{p+1,q+1}$, $\alpha \in \Lambda^{p,q}$

(b) $J = (\sqrt{-1})^{p-q} Id$ on $\Lambda^{p,q}$.

(c) The star operators behave nicely, $*_s : \Lambda^{p,q} \to \Lambda^{n-p,n-q}$.

(d) $*_r : \Lambda^{p,q} \to \Lambda^{n-p,n-q}$, $*_r = *_s J$.

(e) $L^t : \Lambda^{p,q} \to \Lambda^{p-1,q-1}$ because $L^t = *_s L *_s$.

So all the operators behave well as far as bi-degrees are concerned.

5.0.4 Kaehlerian Hodge Theory

Let (X^{2n}, ω) be a compact Kaehler manifold, with $\omega \in \Omega^{1,1}$ a Kaehler form.

From the complex structure we get a mapping $J_p : \Lambda^k(T_p^*) \otimes \mathbb{C} \to \Lambda^k(T_p^*) \otimes \mathbb{C}$. This induces a mapping $J : \Omega^k(X) \to \Omega^k(X)$ by defining $(J\alpha)_p = J_p \alpha_p$ and we have as before the $*$-operators, $*_r, *_s : \Omega^k(X) \to \Omega^{2n-k}$ related by $*_r = *_s \otimes J$.

We also have $\langle , \rangle_r, \langle , \rangle_s$ bilinear forms on Ω^k defined by

$$\langle \alpha, \beta \rangle_r = \int_X \alpha \wedge *_r \bar{\beta} \qquad \langle \alpha, \beta_S = \int_X \alpha \wedge *_s \beta$$

$L : \Omega^k \to \Omega^{k+2}$ is given by $\alpha \mapsto \omega \wedge \alpha$ and $L^t = *_s L *_s = *_r^{-1} L *_r$, the transpose of L with respect to $\langle , \rangle_r$ and $\langle , \rangle_s$.

Finally, we have $d : \Omega^k \to \Omega^{k+1}$ and its transpose $\delta = \delta_r$ the transpose w.r.t. $\langle , \rangle_r$ and δ_s the transpose w.r.t. $\langle , \rangle_s$.

On Ω^k, $\delta_r = (-1)^k *_r^{-1} d *_r$ and $\delta_s = (-1)^k *_s d *_s$. But from $*_r = *_s \circ J$ we get

$$\delta_r = (-1)^k J^{-1} *_s^{-1} d *_s \circ J = J^{-1} \delta_s J$$

We proved a little while ago that $d = [\delta_s, L]$. What happens upon conjugation by J?

$$JdJ^{-1} = [J^{-1}\delta_s J, L] = [\delta, L]$$

We make the following definition

Definition. $d_{\mathbb{C}} = JdJ^{-1}$

So now we have

$$d_{\mathbb{C}} = [\delta, L]$$

Theorem. *d and $d_{\mathbb{C}}$ anti-commute*

We'll prove this later. But for now, we'll prove an important corollary

Corollary. *Let $\Delta = d\delta + \delta d$. Then L and L^t commute with Δ*

Proof. $[d\delta, L] = [d, L]\delta + d[\delta, L]$, and we showed before that $[d, L] = 0$ and $d[\delta, L] = dd_{\mathbb{C}}$. Similarly $[\delta d, L] = d_{\mathbb{C}}d$, so $[\Delta, L] = 0$.

L^t is the Riemannian transpose of L, and in this setting $\Delta^t = \Delta$, so $[\Delta, L^t] = 0$.

We will now use the above to prove Hard Lefshetz

Takef

$$\mathcal{H} = \bigoplus_k \mathcal{H}^k \qquad \mathcal{H}^k = \ker \Delta : \Omega^k \to \Omega^k$$

By the results above $\mathcal{H}$ is invariant under L, L^t and $A = [L, L^t]$. So $\mathcal{H}$ is a finite-dimensional $SL(2, \mathbb{R})$ module.

We prove for $SL(2, \mathbb{R})$ modules that $L^k : \mathcal{H}^{n-k} \to \mathcal{H}^{n+k}$ is bijective.

In the Kaehler case we get the following diagram

$$\begin{array}{ccc} \mathcal{H}^{n-k} & \xrightarrow{L_k} & \mathcal{H}^{n+k} \\ \cong\downarrow & & \downarrow\cong \\ H^{n-k}_{DR}(X) & \xrightarrow{\gamma^k} & H^{n+k}_{DR}(X) \end{array}$$

where $\gamma^k c = [\omega^k] \wedge c$.

Unlike the diagram in the symplectic case, in this case the vertical arrows are bijections. So γ^k is bijective, which is strong Lefshetz.