Lecture 30

Lemma. *$d, d^{\mathbb{C}}$ anti-commute*

Proof. Write $d = \partial + \overline{\partial}$, where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$, $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. Now, $d^{\mathbb{C}} = J^{-1} dJ = J^{-1}\partial J + J^{-1}\overline{\partial} J$. Take $\alpha \in \Omega^{p,q}$ then

$$J^{-1}\partial J\alpha = i^{p-q} J^{-1}\partial\alpha = -\frac{i^{p-q}}{i^{p+1-q}}\partial\alpha = -i\partial\alpha$$
$$J^{-1}\overline{\partial} J\alpha = \frac{i^{p-q}}{i^{p-(q+1)}}\overline{\partial}\alpha = i\overline{\partial}\alpha$$

So $d^{\mathbb{C}} = -i(\partial - \overline{\partial})$, so $d^{\mathbb{C}}, d$ anti-commute because $\partial + \overline{\partial}$ and $\partial = \overline{\partial}$ anti-commute.

Now, some more Hodge Theory.

Take the identity $d^{\mathbb{C}} = [\delta, L]$ and decompose into its homogeneous components, by using $d^{\mathbb{C}} = -i(\partial - \overline{\partial})$. Then $\partial^t : \Omega^{p,q} \to \Omega^{p-1,q}$, $\overline{\partial}^t : \Omega^{p,q} \to \Omega^{p,q-1}$ then $\delta = d^t = \partial^t + \overline{\partial}^t$. So $d^{\mathbb{C}} = [\delta, L]$ because

$$-i(\partial - \overline{\partial}) = [\partial^t, L] + [\overline{\partial}^t, L]$$

and by matching degrees we get

$$i\overline{\partial} = [\partial^t, L] \qquad -\partial = [\overline{\partial}^t, L]$$

We'll play around with these identities for a little while.

We already know that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. And so $(\partial^t)^2 = (\overline{\partial}^t)^2 = \overline{\partial}^t\partial^t + \partial^t\overline{\partial}^t = 0$. Bracket these with L and we get

$$0 = [(\partial^t)^2, L] = [\partial^t, L]\partial^t + \partial^t[\partial^t, L] = i\overline{\partial}\partial^t + \partial^t(i\overline{\partial})$$

so

$$\overline{\partial}\partial^t + \partial^t\overline{\partial} = 0$$

Similarly, from $0 = [(\overline{\partial}^t)^2, L]$ we get

$$\overline{\partial}^t\partial + \partial\overline{\partial}^t = 0$$

Lemma. *The above identities imply the following*

$$\Delta = \Delta_\partial + \Delta_{\overline{\partial}}$$

Proof.

$$\begin{aligned}
\Delta &= dd^t + d^t d \\
&= (\partial + \overline{\partial})(\partial^t + \overline{\partial}^t) + (\partial^t + \overline{\partial}^t)(\partial + \overline{\partial}) \\
&= \Delta_\partial + \Delta_{\overline{\partial}} + (\overline{\partial}\partial^t + \partial\overline{\partial}^t) + (\partial^t\overline{\partial} + \overline{\partial}^t\partial)
\end{aligned}$$

□

Now since $\partial^t\overline{\partial}^t + \overline{\partial}^t\partial^t = 0$ and we get

$$\begin{aligned}
0 &= [\overline{\partial}^t\partial^t + \partial^t\overline{\partial}^t, L] \\
&= [\partial^t\overline{\partial}^t, L] + [\overline{\partial}^t\partial^t, L] \\
&= \partial^t[\overline{\partial}^t, L] + [\partial^t, L]\overline{\partial}^t + \overline{\partial}^t[\partial^t, L] + [\overline{\partial}^t, L]\partial^t \\
&= -i(\partial^t\partial - \overline{\partial}\overline{\partial}^t) - i(\partial\partial^t - \overline{\partial}^t\overline{\partial})
\end{aligned}$$

And we get $\partial^t\partial + \partial\partial^t - \overline{\partial}^t\overline{\partial} - \overline{\partial}\overline{\partial}^t = 0$, i.e.

$$\Delta_\partial - \Delta_{\overline{\partial}} = 0$$

But since $\Delta = \Delta_\partial + \Delta_{\overline{\partial}}$, $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$.

"This has some really neat applications"

Neat Applications

$\Delta_{\overline{\partial}}$ is the Laplace operator for the $\overline{\partial}$ complex

$$\Omega^{1,0} \xrightarrow{\overline{\partial}} \Omega^{i,1} \xrightarrow{\overline{\partial}} \cdots$$

so it maps $\Omega^{i,j}$ to $\Omega^{i,j}$ which implies $\Delta : \Omega^{i,j} \to \Omega^{i,j}$.

So $\mathcal{H}^k = \ker \Delta : \Omega^k \to \Omega^k$ is a direct such

$$\mathcal{H}^k = \bigoplus_{i+j=k} \mathcal{H}^{i,j}$$

where $\mathcal{H}^{i,j} = \mathcal{H}^k \cap \Omega^{i,j}$.

We get a similar decomposition in cohomology

$$H^k(X, \mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}(X) = \operatorname{Im} \mathcal{H}^{i,j}$$

where $\mathcal{H}^{i,j} = \ker \Delta_{\overline{\partial}} : \Omega^{i,j} \to \Omega^{i,j}$, so $\mathcal{H}^{i,j}$ is the jth harmonic space for the Dolbeault complex.

So $H^k(X, \mathbb{C}) = \bigoplus H^{i,j}_{\overline{\partial}}(X)$.