Lecture 32

Properties of the moment map.

For $v, w \in \mathfrak{g}$, we have

$$L_{v_M} d\phi^w = L_{v_M}(\iota(w_M)\omega) = \iota([v_M, w_M])\omega + \iota(w_M) L_{v_M}\omega = \iota([v_M, w_m])\omega = d\phi^{[v,w]}$$

so

$$L_{v_M}\phi^W = \phi^{[v,w]} + \text{constant}$$

Definition. Φ is **equivariant** if and only if

$$L_{v_M}\phi^w = \phi^{[v,w]}$$

Remark: For G abelian, i.e. $[,] = 0$ we have that equivarience implies G invariance, i.e.

$$\Phi(\tau_g(p)) = \Phi(p) \qquad \forall p$$

Also, there is a derivative of the moment map $d\Phi_p : T_pM \to \mathfrak{g}^*$.

Theorem. *(a)* $\operatorname{Im}(d\Phi_p) = \mathfrak{g}_p^\perp$

(b) $\ker d\Phi_p = (T_pG \circ p)^\perp$.

Two parts:

Notation. The "$\perp$" in a) is the the set of all $v \in \mathfrak{g}$ with $\langle v, l\rangle = 0$ for $l \in \operatorname{Im} d\Phi_p$.

The "$\perp$" in b) is the symplectic $\perp$: The set of all $w \in T_pM$ with $\omega_p(w, u) = 0$ for $u \in T_pG \circ p$.

Proof. Recall that $T_pG \circ p = \{v_M(p), v \in \mathfrak{g}\}$. For every $v \in \mathfrak{g}$ and $w \in T_pM$ we have

$$(*) \quad \langle d\Phi_p(w), v\rangle = d\Phi_p^v(w) = \omega_p(v_M, w)$$

Hence if $(*) = 0$ for all w, then $\iota(v_M)\omega_p = 0$, so $v_M(p) = 0$.

Similarly if $(*) = 0$ for all v, then $w \perp T_pG \circ p$. $\square$

De Rham Theory on Quotient Spaces

Let G be a connected Lie group, and τ an action of G on M. Suppose τ is free and proper. Then M/G is a manifold and

$$\pi : M \to M/G = B$$

is a fibration, whose fibers are the G-orbits.

Definition. A k-form $\omega \in \Omega^k(M)$ is **basic** if

(a) It is G-invariant, i.e. $\tau_g^*\omega = \omega$ for all $g \in G$.

(b) $\iota(v_M)\omega = 0$ for all $v \in \mathfrak{g}$.

Theorem. *ω is basic if and only if there exists a $\nu \in \Omega^k(B)$ with $\omega = \pi^*\nu$.*

The proof will be given in a series of lemmas:

Lemma. *For $p \in M$ and $q = \pi(p)$ then sequence*

$$0 \longrightarrow T_pG \circ p \xrightarrow{i} T_pZ \xrightarrow{d\pi_p} T_qB$$

is exact.

Proof. π is a fibration and $G \circ p$ is the fiber through p. N.B. $T_pG \circ p = \{v_M(p), v \in \mathfrak{g}\}$. $\square$

Lemma. *If $\iota(v_M)\mu_p = 0$ for all $v \in \mathfrak{g}$ there exists a $\nu_q \in \Lambda^k(T^*B)$ with $(d\pi_p)^*\nu_q = \mu_p$*

Symplectic Reduction

Assume G is compact, connected and (M, ω) is a symplectic manifold. Let τ be a Hamiltonian action of G with moment map $\Phi : M \to \mathfrak{g}^*$. Assume $0 \in \mathfrak{g}^*$ is a regular value of Φ, i.e. for all $p \in \Phi^{-1}(0)$, $d\Phi_p$ is surjective. Then $Z = \Phi^{-1}(0)$ is a submanifold of M.

Proposition. *Two things*

(a) Z is G-invariant.

(b) The action of G on Z is locally free.

Proof. Z is G-invariant if and only if $\exp tv_M : Z \to Z$ for all $v \in \mathfrak{g}$ if and only if $v_m(p) \in T_pZ$, for all $p \in Z$.

But $v_M(p) \in T_pZ$ if and only if $d\Phi_p(v_M(p)) = 0$ if and only if $d\varphi_p^w(v_M(p)) = 0$ for all w if and only if $L_{v_M}\varphi^w(p) = 0$ on Z if and only if $\varphi^{[v,w]}(p) = 0$ at p. But $p \in \Phi^{-1}(0)$.

To prove that the G action is locally free: At $p \in Z$, $d\Phi_p : T_p \to \mathfrak{g}^*$ is onto. So $(\operatorname{Im} d\Phi_p)^\perp = \mathfrak{g}_p = 0$ if and only if the G action is locally free at p.

Assume G acts free on Z. Since G is compact it acts properly. And $Z/G = M_{red}$ is a C^∞ manifold.

Proposition. *Let $i : Z \to M$ be inclusion and $\pi : Z \to Z/G = M_{red}$. There exists a unique symplectic form ω_{red} on M_{red} with the property that $\iota^*\omega = \pi^*\omega_{red}$. So the orbit space has a god-given symplectic form.*

Proof. $\mu = i^*\omega$, $v \in \mathfrak{g}$, then $\iota(v_Z)\mu = \iota^*(\iota(v_M)\omega) = \iota d\phi^v = 0$, since $\phi^v = 0$ on Z. Moreover, ω G-invariant implies that μ is G-invariant. So we conclude that μ is basic, i.e. $\mu = \pi^*\omega_{red}$, with $\omega_{red} \in \Omega^2(M_{red})$.

Check that this form is symplectic at $p \in M_{red}$, $q = \pi(p), p \in Z$. Then

$$TG \circ p \subset T_pZ = \ker(d\Phi_p) : T_p \to \mathfrak{g}^* = (T_pG \circ p)^\perp$$

But $T_qM_{red} = T_pZ/T_pG \circ p = (T_pG \circ p)^\perp/(T_pG \circ p)$ and we conclude that this is a symplectic vector space. $\square$