Lecture 34

Let G be an n-dimensional compact connected abelian Lie group. Let $\mathfrak{g}$ be the Lie algebra of G.

For an abelian Lie group $\exp : \mathfrak{g} \to G$ is a group epi-morphism and $\mathbb{Z}_G = \ker \exp$ is called the **group lattice** of G. Since exp is an epi-morphisms, $G = \mathfrak{g}/\mathbb{Z}_G$. So we can think of $\exp : \mathfrak{g} \to G$ as a projection $\mathfrak{g} \to \mathfrak{g}/\mathbb{Z}_G$.

Representations of G

We introduce the dual lattice $\mathbb{Z}_G^* \subseteq \mathfrak{g}^*$ a weight lattice, with $\alpha \in \mathfrak{g}^*$ in $\mathbb{Z}_G^*$ if and only if $\alpha(v) \in 2\pi\mathbb{Z}$ for all $v \in \mathbb{Z}_G$.

Suppose we're given $\alpha_i \in \mathbb{Z}^a st_G$, $i = 1, \ldots, d$. We can define a homomorphism $\tau : G \to GL(d, \mathbb{C})$ by

$$(I) \qquad \tau(\exp v)z = (e^{\sqrt{-1}\alpha_1(v)}z_1, \ldots, e^{\sqrt{-1}\alpha_d(v)}z_d)$$

and this is well-defined, because if $v \in \mathbb{Z}_G$, $\tau(\exp v) = 1$. But think of τ as an action of G on $\mathbb{C}^d$. We get a corresponding infinitesimal actions

$$d\tau : \mathfrak{g} \to \mathcal{X}(G) \qquad v \mapsto v_{\mathbb{C}^d} \qquad d\tau(\exp -tv) = \exp tv_{\mathbb{C}^d}.$$

We want a formula for this. We introduce the coordinates $z_i = x_i + \sqrt{-1}y_i$. We claim

$$(II) \qquad v_{\mathbb{C}}d = -\sum \alpha_i(v)\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_j}\right).$$

We must check that for each coordinate z_i

$$\frac{d}{dt}(\tau_{\exp -tv})^* z_i\Big|_{t=0} = L_{v_{\mathbb{C}^d}} z_i.$$

The LHS is

$$\frac{d}{dt}e^{-\sqrt{-1}t\alpha_i(V)}z_i = -\alpha_i(v)z_i$$

and the RHS is

$$\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}\right)(x_i + \sqrt{-1}y_i) = \sqrt{-1}z_i$$

so

$$L_{v_{\mathbb{C}^d}} z_i = \sqrt{-1}\alpha_i(v)z_i$$

Take ω to be the standard kaehler form on $\mathbb{C}^d$

$$\omega = \sqrt{-1}\sum dz_i \wedge d\bar{z}_i = 2\sum dx_i \wedge dy_j$$

Theorem. *τ is a Hamiltonian action with moment map*

$$\Phi : \mathbb{C}^d \to \mathfrak{g}^*$$

where

$$\Phi(z) = \sum |z_i|^2 dz_i$$

Proof.

$$\begin{aligned}\iota(v_{\mathbb{C}^d})\omega &= \left(-\sum \alpha_i(v)\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}\right)\right) \llcorner \sum dx_i \wedge dy_i \\ &= 2\sum \alpha_i(v)x_i dx_i + y_i dy_i = \sum \alpha_i(v) d(x_i^2 + y_i^2) \\ &= d\sum \alpha_i(v)|z_i|^2 = d\langle \Phi, v\rangle\end{aligned}$$

N.B. $\Phi(0) = 0$, $0 \in (\mathbb{C}^d)^G$ implies that Φ is an equivariant moment map. □

Definition. $\alpha_1, \dots, \alpha_d$ are said to be polzarized if for all $v \in \mathfrak{g}$ we have $\alpha_i(v) > 0$.

Theorem. *If $\alpha_1, \dots, \alpha_d$ are polarized then $\Phi : \mathbb{C}^d \to \mathfrak{g}^*$ is proper.*

Proof. The map $\langle \Phi, v\rangle : \mathbb{C}^d \to \mathbb{R}$ is already proper if $\alpha_i(v) > 0$, so the moment map itself is proper. □

Now, given $z \in \mathbb{C}^d$, what can be said about G_z and $\mathfrak{g}_z$?

Notation. $I_z = \{i, z_i \neq 0\}$

Theorem. *(a) $G_z = \{\exp v \mid \alpha_i(v) \in 2\pi\mathbb{Z}$ for all $i \in I_z\}$*

(b) $\mathfrak{g}_z = \{v \mid \alpha_i(v) = 0$ for alli $\in I\}$

Corollary. *τ is locally free at z if and only if span$_{\mathbb{R}}\{\alpha_i, i \in I_z\} = \mathfrak{g}^*$. τ is free at z if and only if span$_{\mathbb{Z}}\{\alpha_i, i \in I_z\} = \mathbb{Z}_G^*$.*

Let $a \in \mathfrak{g}^*$. Is a a regular value of Φ.

Notation.

$$\mathbb{R}_+^d = \{(t_1, \dots, t_d) \in \mathbb{R}^d, t_i \geq 0\}$$

$$I \subset \{1, \dots, d\} \qquad (\mathbb{R}_+^d)_I = \{t \in \mathbb{R}_+^d, t_i > 0 \Leftrightarrow i \in I\}$$

Consider $L : \mathbb{R}_+^d \to \mathfrak{g}^*$

$$L(t) = \sum t_i \alpha_i$$

Assume α_i's are polarized. L is proper. Take $a \in \mathfrak{g}^*$. Let $\Delta_a = L^{-1}(a)$, then Δ_a is a convex polytope. Denote $\mathcal{I}_{\Delta_a} = \{I, (\mathbb{R}_+^d)_I \cap \Delta_a \neq \emptyset\}$. For $I \in \mathcal{I}_\Delta$ we have that $(\mathbb{R}_+^d)_I \cap \Delta =$ the faces of Δ.

Theorem. *$a \in \mathfrak{g}^*$ is a regular value of Φ if and only if for all $I \in \mathcal{I}_{\Delta_a}$ we have span$_{\mathbb{R}}\{a_i, i \in I\} = \mathfrak{g}^*$ and G acts freely on $\Phi^{-1}(a)$ if and only if span$_{\mathbb{Z}}\{a_i, i \in I\} = \mathbb{Z}_G^*$.*

Proof. Φ is the composite of $L : \mathbb{R}_+^d \to \mathfrak{g}^*$ and the map $\gamma : \mathbb{C}^d \to \mathbb{R}_+^d$ which maps $z \mapsto (|z_1|^2, \dots, |z_d|^2)$ so $z \in \Phi^{-1}(a)$ if an only if $\gamma(z) \in \Delta_a$. How just apply above. □

Symplectic Reduction

Take $a \in \mathfrak{g}^*$. Suppose a is a regular value of Φ, i.e. $\mathfrak{g}_z = \{0\}$ for all $z \in \Phi^{-1}(a)$. Then $\mathbb{Z}_a = \Phi^{-1}(a)$ is a compact submanifold of $\mathbb{C}^d$.

Suppose G acts freely on Z_a. Then $M_a = Z_a/G$. Consider $i : Z_a \to \mathbb{C}$, $\pi : Z_a \to M_a$.

Theorem. *There exists a unique symplectic form ω_a on M_a such that $\pi^*\omega_a = i^*\omega_a$.*

Proof. Apply the symplectic quotient procedure to $\Phi^{-1}(a)$. □

Let $G_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}/\mathbb{Z}_G = \mathfrak{g} \otimes \mathbb{C}/\mathbb{Z}_g$. By (I), τ extends to a holomorphic action of $G_{\mathbb{C}}$ on $\mathbb{C}^d$. Then

$$G_{\mathbb{C}} \cdot \Phi^{-1}(a) = \{\tau_g(z) \mid g \in G_{\mathbb{C}}, z \in Z_a\} = \mathbb{C}^d_{\text{stable}}(a)$$

then $M_a = \mathbb{C}^d_{\text{stable}}(a)/G_{\mathbb{C}} =$ the holomorphic description of M_a. ω_a is Kaehler. This M_a is a toric variety.

Theorem.

$$\mathbb{C}^d_{stable}(a) = \bigcup_{I \in \mathcal{I}_\Delta} \mathbb{C}^d_I$$

where

$$\mathbb{C}^d_I = \{z \in \mathbb{C}^d \mid I_z = I\}$$