Lecture 20: Holder continuity of Harmonic functions.

1 Holder continuity of Harmonic functions

In this lecture we will show that harmonic functions need to have a degree of regularity, specifically they must be Holder continuous.

Theorem 1.1 *Let L be a uniformly elliptic operator in divergence form taking*

$$Lu = \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j}. \tag{1}$$

If $u : \mathbb{R}^n \to \mathbb{R}$ is an L harmonic function then u is holder continuous.

The proof is a little involved, so we will first give a sketch of the proof, and then go back to fill in the details. The aim is to use Morrey's lemma.

Proof Pick $x_0 \in \mathbb{R}^n$, and define the operator $\widetilde{L}$ by

$$\widetilde{L} f = \frac{\partial}{\partial x_i} A_{ij}(x_0) \frac{\partial f}{\partial x_j} = A_{ij}(x_0) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{2}$$

Pick $s > 0$, and let v be an L harmonic function with $v = u$ on $\partial B_s(x_0)$. Note that the inequalities we proved in lecture 16 apply to v so, in particular,

$$\int_{B_r(x_0)} |\nabla v|^2 \leq k \left(\frac{r}{s}\right)^n \int_{B_s(x_0)} |\nabla v|^2 \tag{3}$$

for all $r < s$. We use this and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ to estimate

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 &\leq 2\int_{B_r(x_0)} |\nabla v|^2 + 2\int_{B_r(x_0)} |\nabla (u-v)|^2 && (4)\\ &\leq 2k\left(\frac{r}{s}\right)^n \int_{B_s(x_0)} |\nabla v|^2 + 2\int_{B_r(x_0)} |\nabla (u-v)|^2 && (5)\\ &\leq 2k\left(\frac{r}{s}\right)^n \int_{B_s(x_0)} |\nabla v|^2 + 2\int_{B_s(x_0)} |\nabla (u-v)|^2. && (6) \end{aligned}$$

Now use a lemma which we will prove later.

Lemma 1.2 *Let* $||A - A(x_0)|| = \sup_{B_s(x_0),i,j} |A_{ij} - A_{ij}(x_0)|$. *Then*

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{n||A - A(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla v|^2 \tag{7}$$

and

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{n||A - A(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla u|^2 \tag{8}$$

By the first of these we get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(2k\left(\frac{r}{s}\right)^n + 2\left(\frac{n||A - A(x_0)||}{\lambda}\right)^2\right) \int_{B_s(x_0)} |\nabla v|^2. \tag{9}$$

Now we need to estimate this last integral in terms of u. We have

$$\begin{aligned} \int_{B_s(x_0)} |\nabla v|^2 &\leq 2\int_{B_s(x_0)} |\nabla u|^2 + 2\int_{B_s(x_0)} |\nabla(u-v)|^2 && (10) \\ &\leq \left(2 + 2\left(\frac{n||A - A(x_0)||}{\lambda}\right)^2\right) \int_{B_s(x_0)} |\nabla u|^2 && (11) \end{aligned}$$

by lemma 1.2. Plugging this back into **??** gives

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(2k\left(\frac{r}{s}\right)^n + 2\left(\frac{||A - A(x_0)||}{\lambda}\right)^2\right)\left(2 + 2\left(\frac{n||A - A(x_0)||}{\lambda}\right)^2\right) \int_{B_s(x_0)} |\nabla u|^2. \tag{12}$$

By choosing s small we can get $n||A - A(x_0)||$ as small as we like. Therefore, for some constant k' and for any $\delta > 0$ we can pick a small s so that

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(k'\left(\frac{r}{s}\right)^n + \delta\right) \int_{B_s(x_0)} |\nabla u|^2. \tag{13}$$

We need one more lemma.

Lemma 1.3 *Let* ϕ *be a positive and increasing function on the positive reals, and let* α, c *be positive constants. For* $0 < \gamma < \alpha$ *there is* $\delta > 0$ *such that*

$$\phi(r) \leq c_1\left(\left(\frac{r}{s}\right)^\alpha + \delta\right)\phi(s) \tag{14}$$

for $0 < r < s$ *implies*

$$\phi(r) \leq c_2\left(\frac{r}{s}\right)^\gamma \phi(s), \tag{15}$$

where c_2 *is some constant that depends on* c_1, α *and* γ.

In other words for any $0 < \gamma < \alpha$ we can prove **??** by proving **??** for a sufficiently small δ. We will prove this later. Pick $0 < \beta < 1$ and apply this to **??** with $\phi(r) = \int_{B_r(x_0)} |\nabla u|^2$ and $\gamma = n - 2 + 2\beta$ to get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq c \left(\frac{r}{s}\right)^{n-2+2\beta} \int_{B_s(x_0)} |\nabla u|^2. \tag{16}$$

Let $C = \left(\frac{1}{s}\right)^{n-2+2\beta} \int_{B_s(x_0)} |\nabla u|^2$, then

$$\int_{B_r(x_0)} |\nabla u|^2 \leq c \left(\frac{r}{s}\right)^{n-2+2\beta} C, \tag{17}$$

so $u \in C^\beta$ by Morrey's lemma. ∎

Now prove lemma's 1.2 and 1.3.

Lemma 1.2. We wish to show that

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{n||A - A(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla v|^2 \tag{18}$$

and

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{n||A - A(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla u|^2. \tag{19}$$

Proof We will prove the first equation. The proof of the second is analogous. Calculate

$$\lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial(u-v)}{\partial x_i} \frac{\partial(u-v)}{\partial x_j} \tag{20}$$

$$\leq \int_{B_s(x_0)} A_{ij} \frac{\partial(u-v)}{\partial x_i} \frac{\partial u}{\partial x_j} - \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \tag{21}$$

Work on the first term. Clearly $\int_{\partial B_s(x_0)} (u-v) A \nabla u \cdot dS = 0$. By Stokes' theorem we get

$$\int_{B_s(x_0)} A_{ij} \frac{\partial(u-v)}{\partial x_i} \frac{\partial u}{\partial x_j} = -\int_{B_s(x_0)} (u-v) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} = \int_{B_s(x_0)} (u-v) Lu = 0. \tag{22}$$

Plugging this into **??** gives

$$\lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \tag{23}$$

By a similar calculation to **??** we get $\int_{B_s(x_0)} A_{ij}(x_0)\frac{\partial(v-u)}{\partial x_i}\frac{\partial v}{\partial x_j} = 0$, and

$$\begin{aligned} \lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 &\leq \int_{B_s(x_0)} (A_{ij} - A_{ij}(x_0))\frac{\partial(v-u)}{\partial x_i}\frac{\partial v}{\partial x_j} && (24) \\ &\leq ||A - A(x_0)|| \int_{B_s(x_0)} \sum_{i,j} \left|\frac{\partial(v-u)}{\partial x_i}\frac{\partial v}{\partial x_j}\right|. && (25) \end{aligned}$$

Now we need a minilemma, namely that if u, v are n vectors then$\sum_{ij} u_i v_j \leq n|u||v|$. Let w be the vector with $w_i = v_1 + v_2 + \ldots + v_n$ for all i. Note that

$$\begin{aligned} |w| &= \sqrt{n(v_1 + \ldots + v_n)^2} && (26) \\ &\leq n^{3/2}\sqrt{\frac{(v_1 + \ldots + v_n)^2}{n^2}} && (27) \\ &\leq n^{3/2}\sqrt{\frac{v_1^2 + \ldots + v_n^2}{n}} && (28) \\ &\leq n|v| && (29) \end{aligned}$$

since the square of the mean is less than or equal to the mean of the square. From this we get $\sum_{i,j} u_i v_j = u \cdot w \leq |u||w| \leq n|u||v|$ as expected. Applying this to $\nabla(u-v)$ and ∇v gives

$$\begin{aligned} \lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 &\leq n||A - A(x_0)|| \int_{B_s(x_0)} |\nabla(u-v)||\nabla v| && (30) \\ &\leq n||A - A(x_0)|| \left(\int_{B_s(x_0)} |\nabla(u-v)|^2\right)^{1/2} \left(\int_{B_s(x_0)} |\nabla v|^2\right)^{1/2} && (31) \end{aligned}$$

Finally divide and square to get

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{n||A - A(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla v|^2 \tag{32}$$

as required. ∎

Lemma 1.3. We will show that if ϕ is a positive and increasing function on $\mathbb{R}^+$ and

$$\phi(r) \leq c_1 \left(\left(\frac{r}{r'}\right)^\alpha + \delta\right) \phi(s) \tag{33}$$

for $r < r'$ and $0 < \delta < 1$ then

$$\phi(r) \leq c_2(\gamma) \left(\frac{r}{s}\right)^\gamma \phi(s) \tag{34}$$

where $\gamma = \alpha\left(1 + \frac{\log 2c_1}{\log \delta}\right)$, and c_2 is a constant depending on γ.

Proof Choose $\tau = \delta^{1/\alpha}$ so that $\delta = \tau^\alpha$. Then

$$\phi(\tau s) \leq c(\tau^\alpha + \delta)\phi(s) \leq 2c\tau^\alpha \phi(s). \tag{35}$$

Therefore

$$\phi(\tau^k s) \leq (2c_1 \tau^\alpha)^k \phi(s). \tag{36}$$

Pick $\gamma = \alpha \left(1 + \frac{\log 2c_1}{\log \delta}\right)$ so that $2c_1\tau^{\alpha-\gamma} = 1$ and we have

$$\phi(\tau^k s) \leq \tau^{k\gamma}\phi(s). \tag{37}$$

When $r = \tau^k s$ this is precisely what we wanted with $c_2 = 1$. If instead $\tau^{k+1}s \leq r \leq \tau^k s$ then

$$\phi(r) \leq \phi(\tau^k s) \leq \tau^{k\gamma}\phi(s) \leq \frac{1}{\tau}\left(\frac{r}{s}\right)^\gamma \phi(s) \tag{38}$$

which is what we needed. Finally note that by using a small δ we can get γ as close as we like to α (though the constant will become nastier). ∎